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无限长条板中弹性与粘弹性界面裂纹尖端场

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摘　要:　研究无限长条板中粘弹性-弹性界面 Griffith 裂纹在 I型突加载荷作用下, 裂纹尖端动态应力强度因子的时间响应。利用积分变换方法、Fourier 和 Laplace 变换, 分别推导出了弹性和粘弹性问题的控制方程组; 引入位错密度函数, 并结合边界条件, 导出了反映裂纹尖端奇异性的 Cauchy 型奇异积分方程组, 运用 Chebyshev 正交多项式化奇异积分方程组为代数方程组, 用配点法进行求解; 最后用 Laplace 积分变换数值反演方法, 将拉氏域内的解反演到时间域内, 求得动态应力强度因子的时间响应, 并对材料参数的影响进行了分析。结果表明, 剪切松弛参量对 I型动应力强度因子的影响小于对 II型的影响, 而膨胀松弛参量对 I型动应力强度因子的影响大于对 II型的影响。

关键词:　粘弹性; 界面裂纹; 动态应力强度因子; 位错密度函数; 奇异积分方程
中图分类号:　TB332　　**文献标识码**: A

INTERFACE CRACK-TIP FIELD BETWEEN ELASTIC AND VISCOELASTIC MATERIALS IN AN INFINITE LENGTH STRIP

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Abstract:　The dynamic stress intensity factor was studied about the Griffith crack between elastic and visco-elastic layers in an infinite length strip under impact load. The controlling equations expressed by the displacement of elastic and viscoelastic materials were deduced separately after using Laplace and Fourier integral transforms of the basic equations. The displacements in the Laplace domain were obtained by inverse Fourier integral transform; uniting the constitutive and geometrical equations, the analytical expressions of stress in Laplace domain were derived. By defining a dislocation density function, the Cauchy singular integral equations were given according to the boundary conditions and the problem was reduced to algebraic equations using Chebyshev orthogonal polynomial; then it can be solved with the method of collocation dots in the Laplace domain. The time response of the dynamic stress intensity factor was calculated with the numerical method and the influence of material parameters on it was analyzed. The results show that the influence of the shear relaxation parameter on mode I dynamic stress intensity factor is bigger than on mode II, and the influence of the swelling relaxation parameter is opposite.

Keywords:　visco-elasticity; interface crack; dynamic stress intensity factor; dislocation density function; singular integral equation

双材料结构, 如复合金属、复合材料、功能梯度材料等, 在工程实际中的广泛应用推动了界面断裂力学的深入研究。结构界面的力学特性与结构材料的强度、损伤及破坏有着密切联系。实践表明, 界面裂纹是使不同材料连接结构失效的主要原因。因此, 研究两相材料中裂纹等缺陷在界面上产生的应力场有着重要的理论和实际意义。对于工程实际结构, 考虑到有限构形对裂纹尖端渐近应力场的影响, 加之界面材料性质不连续, 使问题复杂化, 很难得到解析解, 需通过解析和数值计算相结合的方法, 以得到符合工程精度的解答。

有关弹性材料界面裂纹的动态响应已有广泛的研

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究，Kundu[1]用 Betti 互等定理研究了界面裂纹在冲击载荷下的动应力强度因子。Li 等[2]用奇异积分方程方法研究了多层板界面裂纹的动态应力强度因子。王晓东等[3]进一步将界面裂纹动态响应问题的研究扩展到带裂尖接触区的修正 Griffith 界面裂纹。Georgiadis[4]对均匀粘弹性体中有限长中心裂纹和币形裂纹在冲击载荷下的动态响应进行了研究。魏培君等[5]对不同粘弹性材料界面裂纹的动态响应问题进行了研究。唐立强等[6~8]对刚性-粘弹性材料及弹性-粘弹性材料界面动态扩展裂纹尖端场进行了渐近分析。

本文作者在此基础上，采用解析和数值相结合的方法对无限长条板中弹性与粘弹性界面裂纹尖端场进行研究，并对材料参数对动应力强度因子的影响进行了分析。图 1 为弹性与粘弹性材料组成的一高为 $2h$ 的无限长条。假设 $y=0$ 平面中有一长为 $2a$ 的裂纹，在 $t=0$ 时刻，裂纹面受 I型冲击载荷作用 $\sigma_{yy}=p_0H(t)$，$H(t)$ 为 Heavyside 单位阶跃函数。

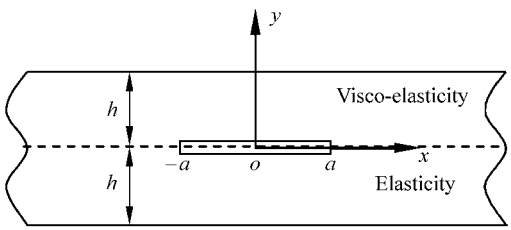

图 1　无限长条板中的 Griffith 裂纹

Fig. 1　A Griffith crack in an infinite strip

1　粘弹性材料的基本方程

粘弹性材料的本构关系采用如下的积分松驰型本构方程：

$$\sigma_{ij}(x,t)=\int_{-\infty}^{t}G_{ijkl}(x,t-\tau)\frac{\partial\varepsilon_{kl}}{\partial\tau}(x,\tau)\,\mathrm{d}\tau \tag{1}$$

其中 G_{ijkl} 为材料的张量松驰函数。假设材料为各向同性的，将式(1)对 t 作 Laplace 变换得

$$\sigma_{ij}^{*}=PG_1^{*}\varepsilon_{ij}^{*}+\frac{P}{3}\varepsilon_{kk}^{*}\delta_{ij}^{*}(G_2^{*}-G_1^{*}) \tag{2}$$

其中：G_1、G_2 为各向同性粘弹性材料的两个独立的松驰函数；$*$ 表示拉式变换；P 为拉式变换参数。小变形下的几何方程和运动方程分别为

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{3}$$

$$\sigma_{ij,j}=\rho\ddot{u}_i \tag{4}$$

在平面应变条件下，由式(2)、(3)和(4)在 Laplace 域内的表达式，得到 Laplace 变换域内位移表示的运动方程，并对其进行 Fourier 变换

$$\begin{cases}-2(2s^2G_1^{*}+s^2G_2^{*}+3\rho P)\overline{u_x^{*}}+\\(2G_2^{*}+G_1^{*})\mathrm{i}s\dfrac{\partial\overline{u_y^{*}}}{\partial y}+3G_1^{*}\dfrac{\partial^2\overline{u_x^{*}}}{\partial y^2}=0,\\-3(s^2G_1^{*}+2\rho P)\overline{u_y^{*}}+(2G_2^{*}+G_1^{*})\mathrm{i}s\dfrac{\partial\overline{u_x^{*}}}{\partial y}+\\2(G_2^{*}+2G_1^{*})\dfrac{\partial^2\overline{u_y^{*}}}{\partial y^2}=0.\end{cases} \tag{5}$$

其中：$\overline{u}$ 表示 u 的傅式变换；s 为傅式变换参数。式(5)的解可以表示如下：

$$\overline{u_x^{*}}(y,P,s)=\sum_{j=1}^{4}m_jF_j(P,s)\mathrm{e}^{n_jy},$$

$$\overline{u_y^{*}}(y,P,s)=\sum_{j=1}^{4}F_j(P,s)\mathrm{e}^{n_jy}. \tag{6}$$

式中：$F_1\sim F_4$ 为未知函数；$n_1\sim n_4$ 和 $m_1\sim m_4$ 分别为

$$n_{1,2}=\pm\sqrt{(s^2G_1^{*}+2\rho P)/G_1^{*}},$$

$$n_{3,4}=\pm\sqrt{[s^2(2G_1^{*}+G_2^{*})+3\rho P]/(2G_1^{*}+G_2^{*})}. \tag{7}$$

$$m_{1,2}=\pm\mathrm{i}\sqrt{(G_1^{*}s^2+2\rho P)/(G_1^{*}s^2)},$$

$$m_{3,4}=\pm\mathrm{i}\sqrt{[s^2(2G_1^{*}+G_2^{*})]/(s^2+3\rho P)}. \tag{8}$$

对式(6)进行 Fourier 逆变换

$$u_x^{*}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}m_jF_j(P,s)\mathrm{e}^{n_jy}\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s,$$

$$u_y^{*}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j(P,s)\mathrm{e}^{n_jy}\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s. \tag{9}$$

由式(9)结合式(2)和式(3)，得到

$$\begin{cases}\sigma_{xx}^{*}=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\sum_{j=1}^{4}A_j(P,s)F_j(P,s)\mathrm{e}^{n_jy}\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s,\\\sigma_{yy}^{*}=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\sum_{j=1}^{4}B_j(P,s)F_j(P,s)\mathrm{e}^{n_jy}\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s,\\\sigma_{xy}^{*}=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\sum_{j=1}^{4}C_j(P,s)F_j(P,s)\mathrm{e}^{n_jy}\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s.\end{cases} \tag{10}$$

其中：$A_j(P,s)$、$B_j(P,s)$、$C_j(P,s)$ 是 P、s 的已知形式，即

$$\begin{cases}A_j(P,s)=\dfrac{P}{3}[(G_2^{*}-G_1^{*})n_j-(2G_1^{*}+G_2^{*})\mathrm{i}sm_j],\\B_j(P,s)=\dfrac{P}{3}[(2G_1^{*}+G_2^{*})n_j-(G_2^{*}-G_1^{*})\mathrm{i}sm_j],\\C_j(P,s)=\dfrac{1}{2}PG_1^{*}(n_jm_j-\mathrm{i}s).\end{cases} \tag{11}$$

2　弹性材料的基本方程

弹性材料的本构方程，小变形下的几何方程和运动方程分别为(在弹性变量的下角标中加 0，以

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区别粘弹性材料)

$$\sigma_{0ij} = 2\mu\varepsilon_{0ij} + \lambda\varepsilon_{0kk}\delta_{ij} \tag{12}$$

$$\varepsilon_{0ij} = \frac{1}{2}(u_{0i,j} + u_{0j,i}) \tag{13}$$

$$\sigma_{0ij,j} = \rho_0 \ddot{u}_{0i} \tag{14}$$

由式(12)、式(13)和式(14)，经 Laplace 变换，可以得到在平面应变、直角坐标系下 Laplace 域内的运动方程

$$\begin{cases} (\lambda+2\mu)\dfrac{\partial^2 u_{0x}^*}{\partial x^2} + (\lambda+\mu)\dfrac{\partial^2 u_{0y}^*}{\partial x\partial y} + \mu\dfrac{\partial^2 u_{0x}^*}{\partial y^2} \\ = \rho_0 P^2 u_{0x}^*, \\ \mu\dfrac{\partial^2 u_{0y}^*}{\partial x^2} + (\lambda+\mu)\dfrac{\partial^2 u_{0x}^*}{\partial x\partial y} + (\lambda+2\mu)\dfrac{\partial^2 u_{0y}^*}{\partial y^2} \\ = \rho_0 P^2 u_{0y}^*. \end{cases} \tag{15}$$

对式(15)进行 Fourier 变换

$$\begin{cases} -[s^2(\lambda+2\mu)+\rho_0 P^2]\overline{u}_{0x}^* + \\ \quad (\lambda+\mu)(\mathrm{i}s)\dfrac{\partial \overline{u}_{0y}^*}{\partial y} + \mu\dfrac{\partial^2 \overline{u}_{0x}^*}{\partial y^2} = 0, \\ -[\mu s^2+\rho_0 P^2]\overline{u}_{0y}^* + (\lambda+\mu)(\mathrm{i}s)\dfrac{\partial \overline{u}_{0x}^*}{\partial y} + \\ \quad (\lambda+2\mu)\dfrac{\partial^2 \overline{u}_{0y}^*}{\partial y^2} = 0. \end{cases} \tag{16}$$

式(16)的解可以表示成

$$\overline{u}_{0x}^*(y,P,s) = \sum_{j=1}^{4} m_{0j}F_{0j}(P,s)\mathrm{e}^{n_{0j}y},$$

$$\overline{u}_{0y}^*(y,P,s) = \sum_{j=1}^{4} F_{0j}(P,s)\mathrm{e}^{n_{0j}y}. \tag{17}$$

式中：$F_{01}\sim F_{04}$ 为未知函数；$n_{01}\sim n_{04}$ 和 $m_{01}\sim m_{04}$ 分别为

$$n_{01,2} = \pm\sqrt{(s^2\mu+\rho_0 P^2)/\mu},$$

$$n_{03,4} = \pm\sqrt{[s^2(\lambda+2\mu)+\rho_0 P^2]/(\lambda+2\mu)}. \tag{18}$$

$$m_{01,2} = \pm\mathrm{i}\sqrt{(\mu s^2+\rho_0 P^2)/(\mu s^2)},$$

$$m_{03,4} = \pm\mathrm{i}\sqrt{(\lambda+2\mu)s^2/[(\lambda+2\mu)s^2+\rho_0 P^2]}. \tag{19}$$

对式(17)进行 Fourier 逆变换

$$u_{0x}^* = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4} m_{0j}F_{0j}(P,s)\mathrm{e}^{n_{0j}y}\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}s,$$

$$u_{0y}^* = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4} F_{0j}(P,s)\mathrm{e}^{n_{0j}y}\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}s. \tag{20}$$

由式(20)结合式(12)和式(13)，得到

$$\begin{cases} \sigma_{0xx}^* = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\sum_{j=1}^{4} A_{0j}(P,s)F_{0j}(P,s)\mathrm{e}^{n_{0j}y}\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}s, \\ \sigma_{0yy}^* = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\sum_{j=1}^{4} B_{0j}(P,s)F_{0j}(P,s)\mathrm{e}^{n_{0j}y}\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}s, \\ \sigma_{0xy}^* = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\sum_{j=1}^{4} C_{0j}(P,s)F_{0j}(P,s)\mathrm{e}^{n_{0j}y}\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}s. \end{cases} \tag{21}$$

其中 $A_{0j}(P,s)$、$B_{0j}(P,s)$、$C_{0j}(P,s)$ 均为 P、s 的已知形式，即

$$\begin{cases} A_{0j}(P,s) = \lambda n_{0j} - (\lambda+2\mu)(\mathrm{i}s)m_{0j}, \\ B_{0j}(P,s) = (\lambda+2\mu)n_{0j} - \lambda(\mathrm{i}s)m_{0j}, \\ C_{0j}(P,s) = \mu(n_{0j}m_{0j} - \mathrm{i}s). \end{cases} \tag{22}$$

3　边界条件和界面连接条件

在边界 $y=\pm h$ 上，正应力和剪应力为零

$$\sigma_{yy}|_{y=h} = 0, \quad \sigma_{0yy}|_{y=-h} = 0,$$

$$\sigma_{xy}|_{y=h} = 0, \quad \sigma_{0xy}|_{y=-h} = 0. \tag{23}$$

在边界 $y=0$ 上，正应力和剪应力是连续的

$$\sigma_{yy}|_{y=0} = \sigma_{0yy}|_{y=0}, \quad \sigma_{xy}|_{y=0} = \sigma_{0xy}|_{y=0}. \tag{24}$$

裂纹所在平面的边界条件为

$$u_x(x,+0) = u_{0x}(x,-0),$$

$$u_y(x,+0) = u_{0y}(x,-0), \quad |x|>a \tag{25}$$

$$\sigma_{yy}(x,\pm 0) = p_0 H(t),$$

$$\sigma_{xy}(x,\pm 0) = 0, \quad |x|<a \tag{26}$$

4　奇异积分方程

为了推导奇异积分方程，引入两个定义在 Laplace 域内的位错密度函数

$$g_1^*(x) = \frac{\partial}{\partial x}[u_x^*(x,+0) - u_{0x}^*(x,-0)],$$

$$g_2^*(x) = \frac{\partial}{\partial x}[u_y^*(x,+0) - u_{0y}^*(x,-0)]. \tag{27}$$

将 u_x^*、u_{0x}^*、u_y^* 和 u_{0y}^* 及附录中公式(1)～式(6)代入上式，并利用 Fourier 变换，可得

$$\int_{-\infty}^{\infty} g_1^*(r)\mathrm{e}^{\mathrm{i}sr}\mathrm{d}r = \mathrm{i}s(f_{11}F_3 + f_{12}F_4),$$

$$\int_{-\infty}^{\infty} g_2^*(r)\mathrm{e}^{\mathrm{i}sr}\mathrm{d}r = \mathrm{i}s(f_{21}F_3 + f_{22}F_4). \tag{28}$$

其中 f_{11}、f_{12}、f_{21}、f_{22} 见附录。

由式(27)和(25)可知，当 $|r|>a$ 时，$g_j(r)=0$。此外，$g_j(r)$ 还必须满足单值条件 $\int_{-a}^{a} g_j(r)\mathrm{d}r = 0$，其中 $j=1,2$。由式(28)可解出未知函数 F_3、F_4，再利用式(10)、式(26)及附录中公式(1)和(2)，当 $|x|<a$ 时，得到

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$$\begin{cases} \lim\limits_{y\to+0}\dfrac{1}{2\pi}\displaystyle\int_{-a}^{a}[h_{11}(x,y,r)g_1^*(r)+ \\ \quad h_{12}(x,y,r)g_2^*(r)]\mathrm{d}r=\dfrac{p_0}{P}, \\ \lim\limits_{y\to+0}\dfrac{1}{2\pi}\displaystyle\int_{-a}^{a}[h_{21}(x,y,r)g_1^*(r)+ \\ \quad h_{22}(x,y,r)g_2^*(r)]\mathrm{d}r=0. \end{cases} \tag{29}$$

其中

$$\begin{cases} h_{11}(x,y,r)=\displaystyle\int_{-\infty}^{\infty}K_{11}(y,s)\mathrm{e}^{\mathrm{i}s(r-x)}\mathrm{d}s, \\ h_{12}(x,y,r)=\displaystyle\int_{-\infty}^{\infty}K_{12}(y,s)\mathrm{e}^{\mathrm{i}s(r-x)}\mathrm{d}s, \\ h_{21}(x,y,r)=\displaystyle\int_{-\infty}^{\infty}K_{21}(y,s)\mathrm{e}^{\mathrm{i}s(r-x)}\mathrm{d}s, \\ h_{22}(x,y,r)=\displaystyle\int_{-\infty}^{\infty}K_{22}(y,s)\mathrm{e}^{\mathrm{i}s(r-x)}\mathrm{d}s. \end{cases} \tag{30}$$

为了确定积分方程奇异行为，需要研究 $x=r$，$y=0$时积分核 K_{ij}的特性。当 $s\to\infty$时，K_{ij}的渐近值为 $K_{11}^{\infty}(x,0,s)=\mathrm{sgn}(s)A_{11}$，$K_{12}^{\infty}(x,0,s)=\mathrm{sgn}(s)A_{12}$，$K_{21}^{\infty}(x,0,s)=\mathrm{sgn}(s)A_{21}$，$K_{22}^{\infty}(x,0,s)=\mathrm{sgn}(s)A_{22}$。加上 K_{ij}并减去其渐近值，代入到式(30)中，则式(29)表示为

$$\begin{cases} \displaystyle\int_{-a}^{a}\left[-\frac{1}{\pi\mathrm{i}}\left(A_{11}\frac{g_1^*(r)}{r-x}+A_{12}\frac{g_2^*(r)}{r-x}\right)+\right. \\ \quad \left.k_{11}(x,0,r)g_1^*(r)+k_{12}(x,0,r)g_2^*(r)\right]\mathrm{d}r=\frac{p_0}{P}, \\ \displaystyle\int_{-a}^{a}\left[-\frac{1}{\pi\mathrm{i}}\left(A_{21}\frac{g_1^*(r)}{r-x}+A_{22}\frac{g_2^*(r)}{r-x}\right)+\right. \\ \quad \left.k_{21}(x,0,r)g_1^*(r)+k_{22}(x,0,r)g_2^*(r)\right]\mathrm{d}r=0. \end{cases} \tag{31}$$

其中

$$k_{ij}(x,0,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[K_{ij}(x,0,s)-K_{ij}^{\infty}(x,0,s)]\mathrm{e}^{\mathrm{i}s(r-x)}\mathrm{d}s,\quad i,j=1,2 \tag{32}$$

由该方程组可以看出，$g_1^*(r)$和 $g_2^*(r)$是耦联在一起的，即使裂纹在 I型载荷条件下，仍然在裂纹前沿形成复合型应力状态。

引入下列无因次变量

$$\chi=r/a,\ r'=x/a,\quad g_i^*(r)=f_i^*(\chi),$$
$$k_{ij}(x,0,r)=L_{ij}(r',0,\chi),\quad i,j=1,2 \tag{33}$$

则当 $|r'|<1$时，式(31)可以表示为无量纲的奇异积分方程组

$$\begin{cases} \displaystyle\int_{-1}^{1}\left[\frac{\mathrm{i}}{\pi}\left(A_{11}\frac{f_1^*(\chi)}{\chi-r'}+A_{12}\frac{f_2^*(\chi)}{\chi-r'}\right)+\right. \\ \quad \left.L_{11}(r',0,\chi)f_1^*(\chi)+L_{12}(r',0,\chi)f_2^*(\chi)\right]\mathrm{d}\chi=\frac{p_0}{P}, \\ \displaystyle\int_{-1}^{1}\left[\frac{\mathrm{i}}{\pi}\left(A_{21}\frac{f_1^*(\chi)}{\chi-r'}+A_{22}\frac{f_2^*(\chi)}{\chi-r'}\right)+\right. \\ \quad \left.L_{21}(r',0,\chi)f_1^*(\chi)+L_{22}(r',0,\chi)f_2^*(\chi)\right]\mathrm{d}\chi=0. \end{cases} \tag{34}$$

奇异积分方程解的单值条件相应写为

$$\int_{-1}^{1}f_j^*(\chi)\mathrm{d}\chi=0,\ j=1,2 \tag{35}$$

方程(34)具有明确的物理意义：$f_j^*(\chi)$为裂纹的位错分布密度函数，积分核对应单个位错的基本解，其中奇异部分表示了裂纹尖端的奇异特性，与静态问题相同，非奇异部分则反映了冲击、材料组合和几何尺寸的影响。

采用 Erdogan[9]的关于奇异积分方程的数值求解方法，未知函数可以表示为

$$f_1^*(\chi)=\sum_{n=0}^{\infty}\left[A_n'T_n(\chi)/\sqrt{1-\chi^2}\right],$$
$$f_2^*(\chi)=\sum_{n=0}^{\infty}\left[B_n'T_n(\chi)/\sqrt{1-\chi^2}\right]. \tag{36}$$

式中，$T_n(\chi)=\cos(n\arccos(\chi))$为第一类 Chebyshev 多项式。因为 $T_0(\chi)=1$，由 $T_n(\chi)$的正交性质[10]，以及式(35)和式(36)，可得 $A_0'=0$，$B_0'=0$。将式(36)代入到式(34)，并利用 $T_n(\chi)$的积分性质[10]，可得

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty}(A_{11}A_n'+A_{12}B_n')U_{n-1}(r')+ \\ \quad \displaystyle\sum_{n=1}^{\infty}\int_{-1}^{1}[A_n'L_{11}(r',0,\chi)+B_n'L_{12}(r',0,\chi)]\times \\ \quad \displaystyle\frac{T_n(\chi)}{\sqrt{1-\chi^2}}\mathrm{d}\chi=\frac{p_0}{P}, \\ \displaystyle\sum_{n=1}^{\infty}(A_{21}A_n'+A_{22}B_n')U_{n-1}(r')+ \\ \quad \displaystyle\sum_{n=1}^{\infty}\int_{-1}^{1}[A_n'L_{21}(r',0,\chi)+B_n'L_{22}(r',0,\chi)]\times \\ \quad \displaystyle\frac{T_n(\chi)}{\sqrt{1-\chi^2}}\mathrm{d}\chi=0,\ |r'|<1. \end{cases} \tag{37}$$

其中，$U_{n-1}(r')=\sin(n\arccos r')/\sqrt{1-r'^2}$为第二类 Chebyshev 多项式。方程组可以通过截断无穷

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级数至有限项，并利用配点法[11]求解。联立求解线性代数方程组(37)，得到未知系数 A_n' 和 B_n'，$n=0, 1, 2, \cdots\cdots$，然后代入式(36)，根据式(33)和式(28)及附录中公式(1)和(2)，可求得 F_1、F_2、F_3、F_4，进一步根据式(10)，可求得 Laplace 域内，粘弹性材料在界面处的应力分布

$$\sigma_{yy}^* = \tilde{B}_4(\tilde{f}_{11}B_1' - \tilde{f}_{21}A_1')x/(\mathrm{i}\tilde{f}_0\sqrt{a^2-x^2}) - \tilde{B}_3(\tilde{f}_{22}A_1' - \tilde{f}_{12}B_1')x/(2\mathrm{i}\tilde{f}_0\sqrt{a^2-x^2}) - \tilde{B}_1\tilde{C}_3(\tilde{f}_{22}A_1' - \tilde{f}_{12}B_1')[\delta(x+\pi) - \delta(x)]/(2\mathrm{i}\tilde{f}_0\tilde{C}_1) \tag{38}$$

$$\sigma_{xy}^* = \frac{\tilde{B}_3\tilde{C}_1(\tilde{f}_{22}A_1' - \tilde{f}_{12}B_1')\tilde{D}}{2\mathrm{i}\tilde{f}_0\tilde{B}_1} - \frac{\tilde{C}_3(\tilde{f}_{22}A_1' - \tilde{f}_{12}B_1')x}{2\mathrm{i}\tilde{f}_0\sqrt{a^2-x^2}} + \frac{\tilde{C}_4(\tilde{f}_{11}B_1' - \tilde{f}_{21}A_1')\tilde{D}}{\mathrm{i}\tilde{f}_0} \tag{39}$$

其中：$\tilde{D} = (x+\pi)[H(-x-\pi) - H(x+\pi)]/2 + x[H(x) - H(-x)]/2$；$\delta(x)$ 为单位脉冲函数；$A_1' = \dfrac{\mathrm{i}p_0A_{22}}{P(A_{12}A_{21} - A_{11}A_{22})}$；$B_1' = \dfrac{p_0A_{21}}{\mathrm{i}P(A_{12}A_{21} - A_{11}A_{22})}$；$A_{11}$、$A_{12}$、$A_{21}$、$A_{22}$、$\tilde{B}_1$、$\tilde{B}_2$、$\tilde{B}_3$、$\tilde{B}_4$、$\tilde{C}_1$、$\tilde{C}_2$、$\tilde{C}_3$、$\tilde{C}_4$、$\tilde{f}_{11}$、$\tilde{f}_{12}$、$\tilde{f}_{21}$、$\tilde{f}_{22}$、$\tilde{f}_0$ 见附录。

为刻化界面裂纹行为，引入 Laplace 变换域内 a 处的复应力强度因子

$$K_{\mathrm{I}}^*(P) + \mathrm{i}K_{\mathrm{II}}^*(P) = \lim_{x\to a}\sqrt{2(x-a)}[\sigma_{yy}^*(x,0) + \mathrm{i}\sigma_{xy}^*(x,0)] \tag{40}$$

通过式(38)、式(39)和式(40)，得到

$$K_{\mathrm{I}}^*(P) = \tilde{B}_3(\tilde{f}_{22}A_1' - \tilde{f}_{12}B_1')\sqrt{a}/(2\mathrm{i}\tilde{f}_0) + \tilde{B}_3(\tilde{f}_{11}B_1' - \tilde{f}_{21}A_1')\sqrt{a}/(\mathrm{i}\tilde{f}_0) \tag{41}$$

$$K_{\mathrm{II}}^*(P) = \tilde{C}_3(\tilde{f}_{22}A_1' - \tilde{f}_{12}B_1')\sqrt{a}/(2\mathrm{i}\tilde{f}_0) \tag{42}$$

为了分析方便，引入无量纲动应力强度因子

$$m_{\mathrm{I}}^*(P) = K_{\mathrm{I}}^*(P)/(p_0\sqrt{\pi a}),$$
$$m_{\mathrm{II}}^*(P) = K_{\mathrm{II}}^*(P)/(p_0\sqrt{\pi a}). \tag{43}$$

其中 $m_{\mathrm{I}}^*(P)$、$m_{\mathrm{II}}^*(P)$ 分别为 I型和 II型动应力强度因子与静应力强度因子的比值。

采用 Georgiadis[12,13]在解决粘弹性动态问题时采用的 Laplace 数值逆变换，可以将 Laplace 变换域内的无量纲动态应力强度因子转换到时间域内，通过引入无量纲时间 $t_{\mathrm{d}} = t/t_0$，其中 $t_0 = \min\{a/\sqrt{\mu_{i0}/\rho_i}, i=1, 2\}$，Laplace 数值逆变换可以表示为

$$m(t_{\mathrm{d}}) = \left(\frac{\mathrm{e}^{ct_{\mathrm{d}}}}{T}\right)\left\{\frac{m^*(c/t_0)}{2t_0} + \sum_{K=1}^{N}\left[\mathrm{Re}\frac{m^*[(c+\mathrm{i}K\pi/T)/t_0]}{t_0}\cos\frac{K\pi t_{\mathrm{d}}}{T}\right] - \sum_{K=1}^{N}\left[\mathrm{Im}\frac{m^*[(c+\mathrm{i}K\pi/T)/t_0]}{t_0}\sin\frac{K\pi t_{\mathrm{d}}}{T}\right]\right\} \tag{44}$$

其中 N、T、c 为满足所需精度而合理选取的参数。

5　奇异积分方程的数值解

令剪切松弛函数和膨胀松弛函数分别为 $G_1(t) = G_{10}\mathrm{e}^{-t/t_1}H(t)$，$G_2(t) = G_{20}\mathrm{e}^{-t/t_2}H(t)$，则 $G_1^*(P) = G_{10}/[P^2 + P(1/t_1)]$，$G_2^*(P) = G_{20}/[P^2 + P(1/t_2)]$。取粘弹性材料参数：$\nu = 0.35$，$G_{10} = 1.8\times10^9$ Pa，$G_{20} = 8.1\times10^9$ Pa，$\rho = 1.4\times10^3$ kg/m^3，$t_1 = t_2 = 3600$ s；弹性材料参数：$\nu = 0.3$，剪切模量 $\mu = 2.5\times10^9$ Pa，$\rho_0 = 2.3\times10^3$ kg/m^3，$\lambda = 3.75\times10^9$ Pa，$h = 1$ m。数值结果如图 2 和图 3 所示。

图 2(a)、2(b)分别是对于不同的 G_{10} 和 G_{20}，弹性与粘弹性材料界面裂纹尖端 K_{I} 的 $m(t_{\mathrm{d}})$-t_{d} 曲线。随着 G_{10} 的减少，曲线峰值降低，并且过冲击现象减弱；随着 G_{20} 的减少，曲线峰值显著降低，并且过冲击现象明显减弱。这说明剪切松弛参量 G_{10} 对动应力强度因子 K_{I} 的影响相对较小，而膨胀松弛参量 G_{20} 对动应力强度因子 K_{I} 的影响较大。

图 2(c)、2(d)分别是对于不同的剪切模量 μ 和拉梅常数 λ，弹性与粘弹性材料界面裂纹尖端 K_{I} 的 $m(t_{\mathrm{d}})$-t_{d} 曲线。随着 μ 值的减少，曲线峰值降低，过冲击现象减弱；随着 λ 值的减少，曲线峰值降低明显，过冲击现象减弱。这说明剪切模量 μ 值对动应力强度因子 K_{I} 的影响小于拉梅常数 λ 值对其的影响。

图 3(a)、3(b)是对于不同的 G_{10} 和 G_{20}，弹性与粘弹性材料界面裂纹尖端 K_{II} 的 $m(t_{\mathrm{d}})$-t_{d} 曲线。随着 G_{10} 的减少，曲线峰值显著降低，并且过冲击现象减弱；随着 G_{20} 的减少，曲线峰值降低，并且过冲击现象减弱。这说明剪切松弛参量 G_{10} 对动应力强度因子 K_{II} 的影响较大，而膨胀松弛参量 G_{20} 对动应力强度因子 K_{II} 的影响较小。

图 3(c)、3(d)是对于不同的剪切模量 μ 和拉梅常数 λ，弹性与粘弹性材料界面裂纹尖端 K_{II}

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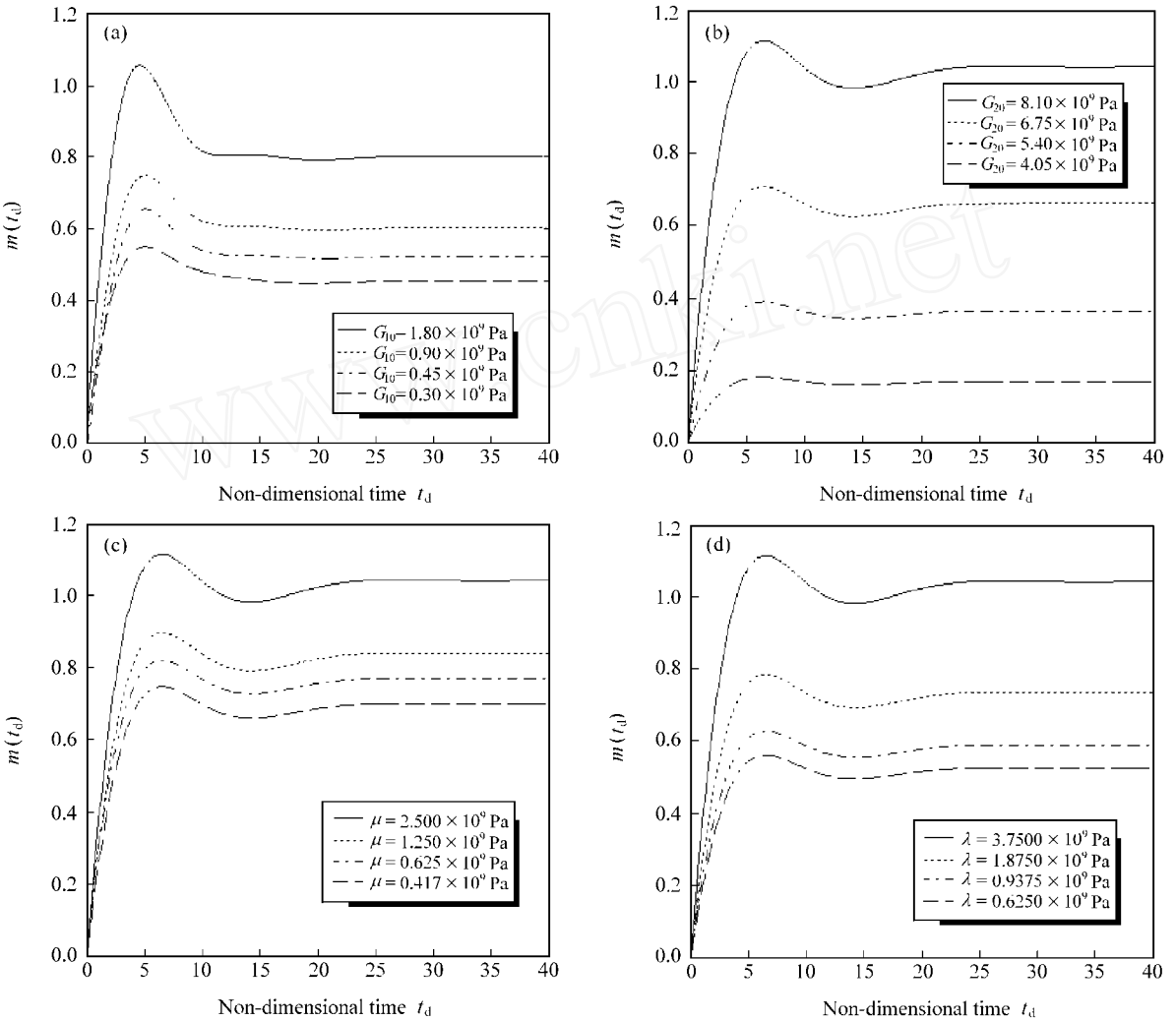

图 2　材料弹性与粘弹性界面裂纹尖端 K_I 的 $m(t_d)$-t_d 曲线

Fig. 2　$m(t_d)$-t_d curves of K_I in the crack-tip of interface between elasticity and visco-elasticity for different parameters

的 $m(t_d)$-t_d 曲线。随着 μ 值的减少，曲线峰值降低，过冲击现象减弱；随着 λ 值的减少，曲线峰值降低较大，过冲击现象减弱。这说明剪切模量 μ 对动应力强度因子 K_{II} 的影响小于拉梅常数 λ 对其的影响。

6　结　论

(1) 剪切松弛参量 G_{10} 对动应力强度因子 K_I 的影响相对较小，而对动应力强度因子 K_{II} 的影响较大。

(2) 膨胀松弛参量 G_{20} 对动应力强度因子 K_I 的影响相对较大，而对动应力强度因子 K_{II} 的影响较小。

(3) 剪切模量 μ 值对动应力强度因子 K_I 和 K_{II} 的影响小于拉梅常数 λ 值对 K_I 和 K_{II} 的影响。

因此，通过对材料参数的合理优化，可以降低结构的动应力强度因子。并且裂纹表面受复杂动载荷作用问题也可使用本方法通过叠加，进行求解。

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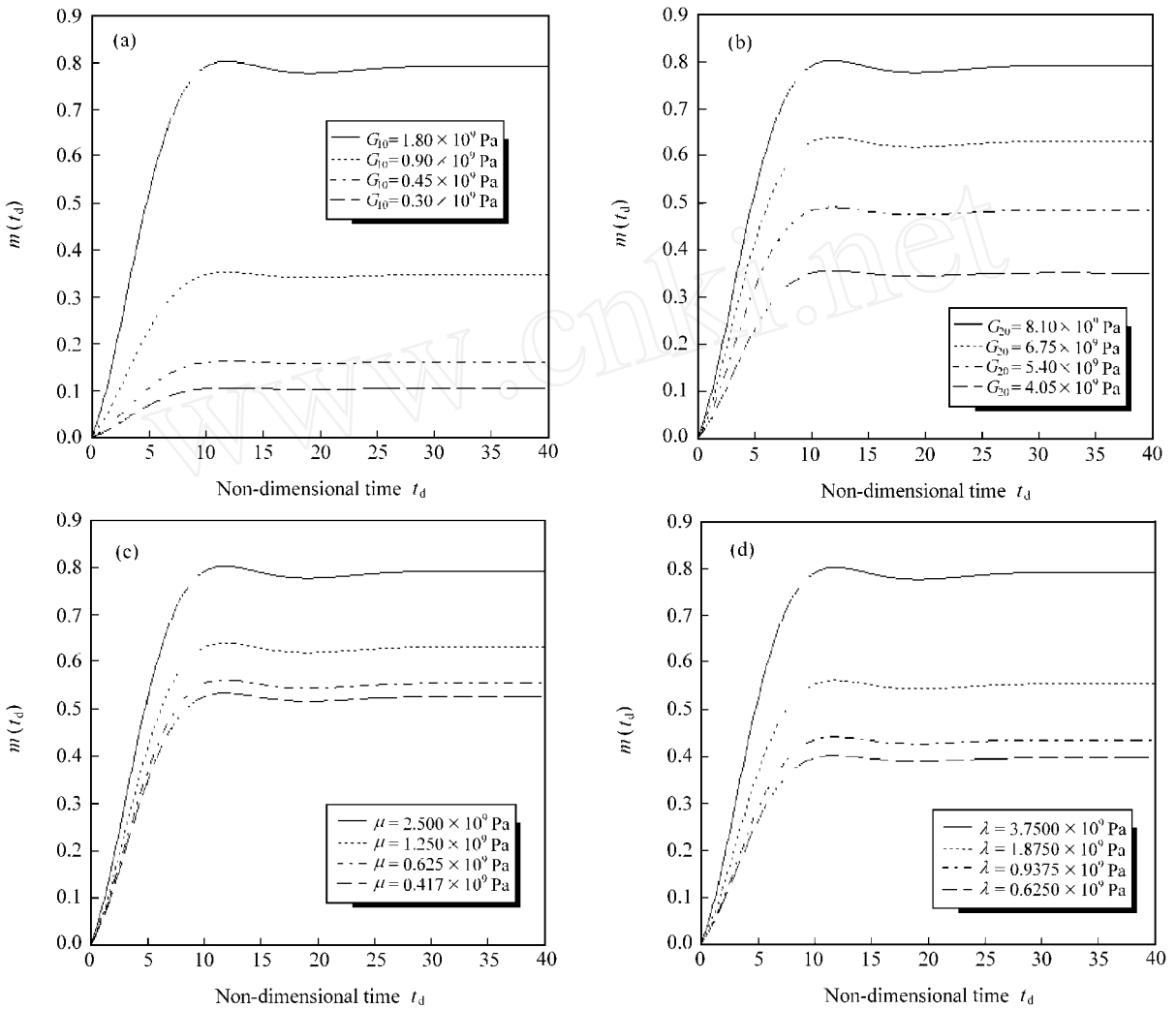

图 3　材料弹性与粘弹性界面裂纹尖端 K_{II}的 $m(t_d)$-t_d 曲线

Fig. 3　$m(t_d)$-t_d curves of K_{II} in the crack-tip of interface between elasticity and visco-elasticity for different parameters

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附 录

$$F_1 = \frac{(B_2 C_3 - B_3 C_2) F_3 e^{n_3 h} + (B_2 C_4 - B_4 C_2) F_4 e^{n_4 h}}{(B_1 C_2 - B_2 C_1) e^{n_1 h}} \tag{1}$$

$$F_2 = \frac{(B_3 C_1 - B_1 C_3) F_3 e^{n_3 h} + (B_4 C_1 - B_1 C_4) F_4 e^{n_4 h}}{(B_1 C_2 - B_2 C_1) e^{n_2 h}} \tag{2}$$

$$F_{01} = \frac{(B_{02} C_{03} - B_{03} C_{02}) F_{03} e^{-n_{03} h} + (B_{02} C_{04} - B_{04} C_{02}) F_{04} e^{-n_{04} h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-n_{01} h}} \tag{3}$$

$$F_{02} = \frac{(B_{03} C_{01} - B_{01} C_{03}) F_{03} e^{-n_{03} h} + (B_{04} C_{01} - B_{01} C_{04}) F_{04} e^{-n_{04} h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-n_{02} h}} \tag{4}$$

$$F_{03} = \Delta_{121} F_3 + \Delta_{122} F_4, \quad F_{04} = \Delta_{123} F_3 + \Delta_{124} F_4 \tag{5}$$

其中

$$\begin{cases} \Delta_{121} = \dfrac{\Delta_{24}\Delta_{11} - \Delta_{21}\Delta_{14}}{\Delta_{24}\Delta_{13} - \Delta_{23}\Delta_{14}}, \quad \Delta_{122} = \dfrac{\Delta_{24}\Delta_{12} - \Delta_{22}\Delta_{14}}{\Delta_{24}\Delta_{13} - \Delta_{23}\Delta_{14}} \\ \Delta_{123} = \dfrac{\Delta_{21}\Delta_{13} - \Delta_{23}\Delta_{11}}{\Delta_{24}\Delta_{13} - \Delta_{23}\Delta_{14}}, \quad \Delta_{124} = \dfrac{\Delta_{22}\Delta_{13} - \Delta_{23}\Delta_{12}}{\Delta_{24}\Delta_{13} - \Delta_{23}\Delta_{14}} \end{cases} \tag{6}$$

其中

$$\Delta_{11} = \frac{B_1 (B_2 C_3 - B_3 C_2) e^{(n_2 + n_3) h} + B_2 (B_3 C_1 - B_1 C_3) e^{(n_1 + n_3) h} + B_3 (B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}}{(B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}} \tag{7}$$

$$\Delta_{12} = \frac{B_1 (B_2 C_4 - B_4 C_2) e^{(n_2 + n_4) h} + B_2 (B_4 C_1 - B_1 C_4) e^{(n_1 + n_4) h} + B_4 (B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}}{(B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}} \tag{8}$$

$$\Delta_{13} = \frac{B_{01} (B_{02} C_{03} - B_{03} C_{02}) e^{-(n_{02} + n_{03}) h} + B_{02} (B_{03} C_{01} - B_{01} C_{03}) e^{-(n_{01} + n_{03}) h} + B_{03} (B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}} \tag{9}$$

$$\Delta_{14} = \frac{B_{01} (B_{02} C_{04} - B_{04} C_{02}) e^{-(n_{02} + n_{04}) h} + B_{02} (B_{04} C_{01} - B_{01} C_{04}) e^{-(n_{01} + n_{04}) h} + B_{04} (B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}} \tag{10}$$

$$\Delta_{21} = \frac{C_1 (B_2 C_3 - B_3 C_2) e^{(n_2 + n_3) h} + C_2 (B_3 C_1 - B_1 C_3) e^{(n_1 + n_3) h} + C_3 (B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}}{(B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}} \tag{11}$$

$$\Delta_{22} = \frac{C_1 (B_2 C_4 - B_4 C_2) e^{(n_2 + n_4) h} + C_2 (B_4 C_1 - B_1 C_4) e^{(n_1 + n_4) h} + C_4 (B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}}{(B_1 C_2 - B_2 C_1) e^{(n_1 + n_2) h}} \tag{12}$$

$$\Delta_{23} = \frac{C_{01} (B_{02} C_{03} - B_{03} C_{02}) e^{-(n_{02} + n_{03}) h} + C_{02} (B_{03} C_{01} - B_{01} C_{03}) e^{-(n_{01} + n_{03}) h} + C_{03} (B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}} \tag{13}$$

$$\Delta_{24} = \frac{C_{01} (B_{02} C_{04} - B_{04} C_{02}) e^{-(n_{02} + n_{04}) h} + C_{02} (B_{04} C_{01} - B_{01} C_{04}) e^{-(n_{01} + n_{04}) h} + C_{04} (B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-(n_{01} + n_{02}) h}} \tag{14}$$

$$F_{01} = \Delta_{111} F_3 + \Delta_{112} F_4, \quad F_{02} = \Delta_{113} F_3 + \Delta_{114} F_4 \tag{15}$$

其中

$$\Delta_{111} = \frac{(B_{02} C_{03} - B_{03} C_{02}) \Delta_{121} e^{-n_{03} h} + (B_{02} C_{04} - B_{04} C_{02}) \Delta_{123} e^{-n_{04} h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-n_{01} h}} \tag{16}$$

$$\Delta_{112} = \frac{(B_{02} C_{03} - B_{03} C_{02}) \Delta_{122} e^{-n_{03} h} + (B_{02} C_{04} - B_{04} C_{02}) \Delta_{124} e^{-n_{04} h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-n_{01} h}} \tag{17}$$

$$\Delta_{113} = \frac{(B_{03} C_{01} - B_{01} C_{03}) \Delta_{121} e^{-n_{03} h} + (B_{04} C_{01} - B_{01} C_{04}) \Delta_{123} e^{-n_{04} h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-n_{02} h}} \tag{18}$$

$$\Delta_{114} = \frac{(B_{03} C_{01} - B_{01} C_{03}) \Delta_{122} e^{-n_{03} h} + (B_{04} C_{01} - B_{01} C_{04}) \Delta_{124} e^{-n_{04} h}}{(B_{01} C_{02} - B_{02} C_{01}) e^{-n_{02} h}} \tag{19}$$

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$$f_{11} = \frac{m_1(B_2C_3 - B_3C_2)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_1h}} + \frac{m_2(B_3C_1 - B_1C_3)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_2h}} + m_3 - m_{01}\Delta_{111} - m_{02}\Delta_{113} - m_{03}\Delta_{121} - m_{04}\Delta_{123} \tag{20}$$

$$f_{12} = \frac{m_1(B_2C_3 - B_3C_2)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_1h}} + \frac{m_2(B_4C_1 - B_1C_4)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_2h}} + m_4 - m_{01}\Delta_{112} - m_{02}\Delta_{114} - m_{03}\Delta_{122} - m_{04}\Delta_{124} \tag{21}$$

$$f_{21} = \frac{(B_2C_3 - B_3C_2)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_1h}} + \frac{(B_3C_1 - B_1C_3)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_2h}} + 1 - \Delta_{111} - \Delta_{113} - \Delta_{121} - \Delta_{123} \tag{22}$$

$$f_{22} = \frac{(B_2C_3 - B_3C_2)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_1h}} + \frac{(B_4C_1 - B_1C_4)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_2h}} + 1 - \Delta_{112} - \Delta_{114} - \Delta_{122} - \Delta_{124} \tag{23}$$

积分核 K_{ij} **为**

$$\begin{cases} K_{11}(y,s) = \dfrac{(Q_{11}f_{22} - Q_{12}f_{21})}{\mathrm{i}s(f_{22}f_{11} - f_{21}f_{12})}, \quad K_{12}(y,s) = \dfrac{(Q_{12}f_{11} - Q_{11}f_{12})}{\mathrm{i}s(f_{22}f_{11} - f_{21}f_{12})} \\ K_{21}(y,s) = \dfrac{(Q_{21}f_{22} - Q_{22}f_{21})}{\mathrm{i}s(f_{22}f_{11} - f_{21}f_{12})}, \quad K_{22}(y,s) = \dfrac{(Q_{22}f_{11} - Q_{21}f_{12})}{\mathrm{i}s(f_{22}f_{11} - f_{21}f_{12})} \end{cases} \tag{24}$$

其中

$$Q_{11} = \frac{B_1(B_2C_3 - B_3C_2)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_1h}}e^{n_1y} + \frac{B_2(B_3C_1 - B_1C_3)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_2h}}e^{n_2y} + B_3e^{n_3y} \tag{25}$$

$$Q_{12} = \frac{B_1(B_2C_4 - B_4C_2)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_1h}}e^{n_1y} + \frac{B_2(B_4C_1 - B_1C_4)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_2h}}e^{n_2y} + B_4e^{n_4y} \tag{26}$$

$$Q_{21} = \frac{C_1(B_2C_3 - B_3C_2)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_1h}}e^{n_1y} + \frac{C_2(B_3C_1 - B_1C_3)e^{n_3h}}{(B_1C_2 - B_2C_1)e^{n_2h}}e^{n_2y} + C_3e^{n_3y} \tag{27}$$

$$Q_{22} = \frac{C_1(B_2C_4 - B_4C_2)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_1h}}e^{n_1y} + \frac{C_2(B_4C_1 - B_1C_4)e^{n_4h}}{(B_1C_2 - B_2C_1)e^{n_2h}}e^{n_2y} + C_4e^{n_4y} \tag{28}$$

$$A_{11} = \frac{\{6(\lambda+\mu)(G_2 - G_1)(1 - \sqrt{2G_1 + G_2}) - P(G_1 + 2G_2)[2G_1 + G_2 + (G_2 - G_1)\sqrt{2G_1 + G_2}]\}P}{3[6(\lambda+\mu)(\sqrt{2G_1 + G_2} - 1) + P(G_1 + 2G_2)(1 + \sqrt{2G_1 + G_2})]} \tag{29}$$

$$A_{12} = \frac{\mathrm{i}\{6(\lambda+\mu)(2G_1 + G_2)(-1 + \sqrt{2G_1 + G_2}) + P(G_1 + 2G_2)[2G_1 + G_2 + (G_2 - G_1)\sqrt{2G_1 + G_2}]\}P}{3[6(\lambda+\mu)(\sqrt{2G_1 + G_2} - 1) + P(G_1 + 2G_2)(1 + \sqrt{2G_1 + G_2})]} \tag{30}$$

$$A_{21} = \frac{\mathrm{i}PG_1(\sqrt{2G_1 + G_2} - 1)[6(\lambda+\mu) + P(G_1 + 2G_2)]}{2[6(\lambda+\mu)(-1 + \sqrt{2G_1 + G_2}) + P(G_1 + 2G_2)(1 + \sqrt{2G_1 + G_2})]} \tag{31}$$

$$A_{22} = \frac{PG_1(\sqrt{2G_1 + G_2} - 1)[-6(\lambda+\mu) + P(G_1 + 2G_2)]}{2[6(\lambda+\mu)(-1 + \sqrt{2G_1 + G_2}) + P(G_1 + 2G_2)(1 + \sqrt{2G_1 + G_2})]} \tag{32}$$

$$\begin{cases} \tilde{B}_1(P) = \dfrac{P}{3}(G_1^* + 2G_2^*), \ \tilde{B}_4(P) = -\dfrac{P}{3}[(2G_1^* + G_2^*) + (G_2^* - G_1^*)\sqrt{2G_1^* + G_2^*}] \\ \tilde{B}_2(P) = -\dfrac{P}{3}(G_1^* + 2G_2^*), \ \tilde{B}_3(P) = \dfrac{P}{3}[(2G_1^* + G_2^*) + (G_2^* - G_1^*)\sqrt{2G_1^* + G_2^*}] \end{cases} \tag{33}$$

$$\tilde{C}_1(P) = \tilde{C}_2(P) = \mathrm{i}P\sqrt{P\rho G_1^*}, \ \tilde{C}_3(P) = \tilde{C}_4(P) = \mathrm{i}PG_1^*(\sqrt{2G_1^* + G_2^*} - 1)/2 \tag{34}$$

$$\tilde{B}_{01}(P) = \tilde{B}_{03}(P) = 2(\lambda+\mu), \ \tilde{B}_{02}(P) = \tilde{B}_{04}(P) = -2(\lambda+\mu) \tag{35}$$

$$\tilde{C}_{01}(P) = \tilde{C}_{02}(P) = P\mu\sqrt{2\rho_0/\mu}, \ \tilde{C}_{03}(P) = \tilde{C}_{04}(P) = P\mu\sqrt{\rho_0/(\lambda+2\mu)} \tag{36}$$

$$\tilde{f}_0 = \frac{\tilde{C}_3}{\tilde{C}_1}\left[\frac{(\tilde{m}_4 - \tilde{m}_2)}{2} + \frac{(\tilde{m}_{01} - \tilde{m}_4)(\tilde{C}_1\tilde{B}_{01} + \tilde{B}_2)}{2\tilde{B}_{01}\tilde{C}_{01}(1 + \tilde{C}_{01})} + \frac{(\tilde{m}_{03} - \tilde{m}_4)\tilde{C}_{01}(\tilde{B}_2 + \tilde{C}_1B_{02})}{\tilde{B}_{01}(\tilde{C}_{01} - \tilde{C}_{03})(1 + \tilde{C}_{01})}\right] \tag{37}$$

$$\tilde{f}_{11} = -\frac{\tilde{C}_3}{\tilde{C}_1}\left[\frac{\tilde{m}_2}{2} + \frac{\tilde{m}_{01}(\tilde{C}_{03}\tilde{B}_1 + \tilde{B}_{01}\tilde{C}_1) - \tilde{m}_{03}(\tilde{B}_1\tilde{C}_{01} + \tilde{B}_{01}\tilde{C}_1)}{2\tilde{B}_{01}(\tilde{C}_{03} - \tilde{C}_{01})}\right] \tag{38}$$

$$\tilde{f}_{12} = -\frac{\tilde{C}_4}{\tilde{C}_1}\left[\frac{\tilde{m}_2}{2} + \frac{\tilde{m}_{01}(\tilde{B}_1\tilde{C}_{03} - \tilde{B}_{01}\tilde{C}_1) - \tilde{m}_{03}(\tilde{B}_1\tilde{C}_{01} - \tilde{B}_{01}\tilde{C}_1)}{2\tilde{B}_{01}(\tilde{C}_{03} - \tilde{C}_{01})}\right] \tag{39}$$

$$\tilde{f}_{21} = -\tilde{C}_3(\tilde{B}_{01} + \tilde{B}_1)/(2\tilde{C}_1\tilde{B}_{01}), \ \tilde{f}_{22} = -\tilde{C}_4(\tilde{B}_{01} + \tilde{B}_1)/(2\tilde{C}_1\tilde{B}_{01}) \tag{40}$$

$$\begin{cases} \tilde{m}_1 = \mathrm{i}, \ \tilde{m}_2 = -\mathrm{i}, \ \tilde{m}_3 = \mathrm{i}\sqrt{2G_1^* + G_2^*}, \ \tilde{m}_4 = -\mathrm{i}\sqrt{2G_1^* + G_2^*} \\ \tilde{m}_{01} = \mathrm{i}, \ \tilde{m}_{02} = -\mathrm{i}, \ \tilde{m}_{03} = \mathrm{i}, \ \tilde{m}_{04} = -\mathrm{i} \end{cases} \tag{41}$$

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