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压电梯度薄壳的高阶理论解

伍晓红, 沈亚鹏, 陈常青, 田晓耕
(西安交通大学 建筑与力学学院, 西安 710049)

摘 要: 压电功能梯度执行器能产生较大的位移、降低应力峰值并避免了粘结层带来的问题, 压电梯度超声换能器能拓展频带宽度。本文作者提出了一个简单而有效的求解压电梯度薄壳力、电行为特性的高阶理论。设定位移分量为壳厚的线性函数, 而电势沿厚度方向为二次分布。考虑了压电作动元的驱动信号不同时所具有的不同形式的电荷平衡方程。应用 Fourier 级数法得到压电系数沿厚度坐标变化的梯度壳的力电耦合的解析解。所得结果可退化至梁、板等多种特殊情况。利用所得方程分析了一非均匀简支压电层合板, 并与三维精确结果作了对比, 两者吻合得很好, 表明该理论的正确性。最后具体求解了压电梯度圆柱壳的力、电特性, 给出了位移、应力、电势沿厚度方向的变化规律。

关键词: 高阶理论; 功能梯度; 压电壳
中图分类号: TB330.1 **文献标识码:** A

HIGH ORDER THEORY SOLUTIONS OF FUNCTIONALLY GRADED PIEZOELECTRIC SHELLS

WU Xiaohong, SHEN Yapeng, CHEN Changqing, TIAN Xiaogeng
(School of Civil Engineering and Mechanics, Xi'an Jiaotong University, Xi'an 710049, China)

Abstract: A new type of actuator using functionally graded piezoelectric ceramics can produce large out of plane displacements, reduce mid-plane stresses and avoid failure from internal debonding or from stress peaks. A functionally graded piezoelectric ultrasound transducer has a broadband frequency characteristic. This study presents a simple and accurate high order theory to examine the electromechanical behaviour of piezoelectric generic shells with thickness-graded material properties. The displacement components are assumed to be a linear function of the shell thickness, while the electric potential is taken as a quadratic distribution through the thickness coordinate. Both equations of motion and boundary conditions of the shells are simultaneously obtained using the variational principle. Different types of charge equilibrium equations are considered, which correspond to the respective driving power when it is acted as actuators. The Fourier series method is then applied to obtain the analytical solution of mechanical and electric fields of functionally graded piezoelectric shells. These newly derived equations can be reduced to many typical structures as beam, plate and circular cylindrical shell. In terms of analyzing a simply-supported inhomogeneous and laminated piezoelectric plate, numerical results of the proposed formulation coincide well with exact solutions in previous literature. Finally, numerical studies are performed to examine the electromechanical responses of functionally graded piezoelectric circular cylindrical shells. The effects of graded material properties on the displacement, stresses and electric potential are clearly exhibited. This piece of work was motivated by the increased general use of functionally graded materials and piezoelectric materials and also a need to understand the electromechanical behaviour of functionally graded piezoelectric materials.

Key words: high order theory; functionally graded; piezoelectric shells

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作者介绍: 伍晓红(1971-), 女, 博士研究生, 讲师, 主要研究方向: 功能梯度材料特性。
通讯作者: 沈亚鹏, E-mail: ypshen@mail.edu.cn

压电材料由于其特有的正、逆压电效应在各个领域特别是智能结构中已得到广泛的应用, 如用作作动器、传感器, 用于结构的振动控制和形状控制等。但一些压电元件中压电层的层间呈现应力集中且粘结处可能出现蠕变、开裂和脱落等现象而导致器件应用范围受限且易失效。近年来, 功能梯度材料(FGM)越来越多地引起人们的极大兴趣和关注, 这类材料在制备过程中连续地控制其微观要素, 使材料内部不存在明显的界面, 可缓和材料内部的热应力, 成为应用于高温环境的新一代功能材料。人们将功能梯度的思想推广至压电材料, 即使压电材料性能参数沿某一方向连续变化来得到期望的性能。压电梯度材料可用于弯曲作动器或超声换能器等, 如已有的独石型压电梯度陶瓷执行器[1]和压电梯度双执行器[2]与传统的执行器相比, 能产生较大的位移, 降低应力峰值并避免了粘结层带来的问题。Yamada[3]等通过在压电陶瓷板中形成温度梯度, 制造了能拓展频带宽度的压电梯度换能器。

压电智能结构对力电荷载的响应已有大量研究, 但由于得到三维精确解的复杂性, 主要集中于对简单形状如简支板、圆柱壳和压电层合板(含热压电板)的研究。而对复杂的压电结构, 常采用近似而有效的分析模型[4~5]。目前对压电功能梯度结构的研究又悄然兴起。Reddy 和 Cheng[6]用渐进展开法结合传递矩阵公式得到了智能功能梯度(FG)结构(由弹性 FGM 层和压电层组成)的渐进解。Huang[7]用有限元方法分析了智能功能梯度压电结构的热、电响应。Chen 和 Ding[8]基于状态方程分析了压电功能梯度板的弯曲问题。但有关压电梯度壳的传感和作动的力电行为的分析还很少见到。

本文作者提出了一个简单而有效的求解压电梯度薄壳的力、电行为特性的高阶理论。设定位移分量和电势为壳厚的函数, 考虑了压电作动元的驱动信号不同时所具有的不同形式的电荷平衡方程。应用 Fourier 级数法得到压电系数沿厚度坐标变化的梯度壳的力电响应的解析解。所得结果可退化至梁、板等多种特殊情况。根据所得方程讨论了一非均匀压电层合板实例, 并与三维精确结果作了对比, 两者很好地吻合, 表明该理论的正确性。最后具体求解了压电梯度圆柱壳的力、电特性, 给出了位移、应力、电势、电位移沿厚度方向的变化规律, 并与均匀压电圆柱壳相比较, 显示了材料常数梯度变化对响应的影响。

1 压电薄壳的基本方程

由于梁、板、圆柱壳等常见结构都可以认为是薄壳的特殊情况, 这里讨论的是图 1 所示的具有一般性的厚度为 2h 的梯度薄壳结构。为方便起见, 采用了曲面坐标 T_1, T_2, T_3。其中 T_1, T_2 是中面正交的主曲率线。A_1, A_2, R_1, R_2 分别为对应的拉梅系数和主曲率半径, 在假定厚度较小的情况下(即 $hn \ll R_1, hn \ll R_2$), 可认为它们与横向坐标无关的(在以下的公式中, z 和 T_3 不加区分)。

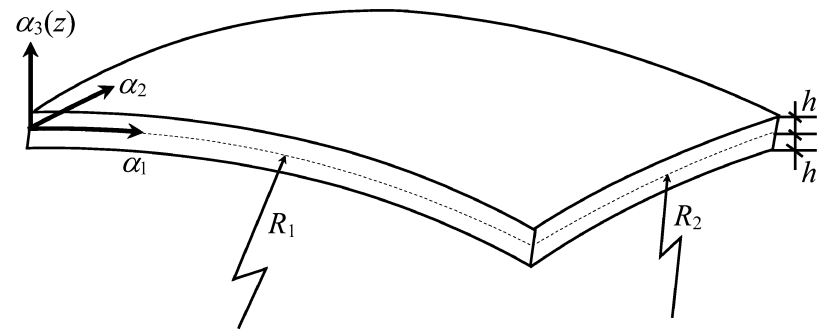

图 1 双曲率薄壳的示意图

Fig. 1 Configuration of a piezoelectric generic shell

线性压电材料的本构方程为

$$e_{ij} = C_{ijkl}X_{kl} - e_{kij}E_k$$
$$D_i = e_{ikl}X_{kl} + \in_{ik}E_k \tag{1}$$

材料参数仅沿厚度方向梯度变化

$$C_{ijkl} \equiv C_{ijkl}(z);\ e_{kij} \equiv e_{kij}(z);\ \in_{ik} \equiv \in_{ik}(z) \tag{2}$$

由壳体的一般理论, (1) 式中的应变分量和电场强度分量与位移分量 $u_i(i=1,2,3)$ 和电势 h 的关系为

$$X_{11} = \frac{\partial u_1}{A_1\partial T_1} + u_2\frac{\partial A_1}{A_1A_2\partial T_2} + \frac{u_3}{R_1};$$

$$X_{22} = \frac{\partial u_2}{A_2\partial T_2} + u_1\frac{\partial A_2}{A_1A_2\partial T_1} + \frac{u_3}{R_2};$$

$$X_{33} = \frac{\partial u_3}{\partial T_3};$$

$$V_{23} = \frac{\partial u_2}{\partial T_3} + \frac{\partial u_3}{A_2\partial T_2} - \frac{u_2}{R_2};$$

$$V_{31} = \frac{\partial u_1}{\partial T_3} + \frac{\partial u_3}{A_1\partial T_1} - \frac{u_1}{R_1};$$

$$V_{12} = \frac{\partial u_1}{A_2\partial T_2} + \frac{\partial u_2}{A_1\partial T_1} - u_2\frac{\partial A_2}{A_1A_2\partial T_1} - u_1\frac{\partial A_1}{A_1A_2\partial T_2};$$

$$V_{ij} = 2X_{ij} \quad (i \neq j) \tag{3}$$

$$E_1 = -\frac{\partial h}{A_1\partial T_1};\ E_2 = -\frac{\partial h}{A_2\partial T_2};\ E_3 = -\frac{\partial h}{\partial T_3} \tag{4}$$

对表面作用有力载荷 T_i 和电荷面密度 w 的压电介质, 由哈密顿变分原理可得

$$W\int_{t_0}^{t_1}dt\left(\int_V(K - H)dV - \right.$$

$$\int_{S_e} T_i \delta u_i dS + \int_{S_w} w \delta h dS) = 0 \qquad (5)$$

式中: K 为动能密度, $K = \rho \dot{u}_i \dot{u}_i / 2$; H 为电焓密度, $H = e_{ij} X_{ij} / 2 - D_i E_i / 2$。

由本构方程(1)可得: $e_{ij} = \partial H / \partial X_{ij}$; $D_i = -\partial H / \partial E_i$。

将 K, H, e_{ij}, D_i 的表达式代入(5)式, 得

$$\int_{t_0}^{t_1} dt \left(\int_V (e_{ij} \delta X_{ij} + \rho \ddot{u}_i \delta u_i + D_i \delta E_i) dV - \int_{S_e} T_i \delta u_i dS + \int_{S_w} w \delta h dS \right) = 0 \qquad (6)$$

则

$$\int_V e_{kl} \delta X_{kl} dV = - \int_V \rho \ddot{u}_i \delta u_i dV + \int_{S_e} T_i \delta u_i dS \qquad (7)$$

$$\int_V D_k \delta E_k dV = - \int_{S_w} w \delta h dS \qquad (8)$$

从而可以独立地研究压电结构的力平衡和电荷平衡方程。

根据一阶剪切变形理论, 并考虑横向正应变的影响, 设定位移、电势按如下规律分布

$$u_1(T_1, T_2, T_3) = u_1^{(0)}(T_1, T_2) + z u_1^{(1)}(T_1, T_2);$$

$$u_2(T_1, T_2, T_3) = u_2^{(0)}(T_1, T_2) + z u_2^{(1)}(T_1, T_2);$$

$$u_3(T_1, T_2, T_3) = u_3^{(0)}(T_1, T_2) + z u_3^{(1)}(T_1, T_2);$$

$$h(T_1, T_2, T_3) = h^{(0)}(T_1, T_2) + z h^{(1)}(T_1, T_2) + z^2 h^{(2)}(T_1, T_2) \qquad (9)$$

式中: $u_3^{(1)}$ 的引入是为了描述压电结构作为作动元件时, 由外加电压或电荷产生的横向正应变; $h^{(2)}$ 则是为了描述压电传感片或热压电元中电势沿横向的抛物线分布规律。我们的研究[9~10]发现: 当压电结构承受力、热荷载而电荷载为零时, 其电势沿厚度方向按抛物线分布; 当压电结构作为作动片时, 其电势沿厚度方向分布则仍是近于线性的。

将(9)式代入(3)式和(4)式得应变和电场强度的表达式为

$$X_{11} = X_{11}^{(0)} + z X_{11}^{(1)}; X_{22} = X_{22}^{(0)} + z X_{22}^{(1)}; X_{33} = X_{33}^{(0)};$$

$$V_{23} = V_{23}^{(0)} + z V_{23}^{(1)}; \; V_{31} = V_{31}^{(0)} + z V_{31}^{(1)};$$

$$V_{12} = V_{12}^{(0)} + z V_{12}^{(1)} \qquad (10)$$

$$E_1 = E_1^{(0)} + z E_1^{(1)} + z^2 E_1^{(2)};$$

$$E_2 = E_2^{(0)} + z E_2^{(1)} + z^2 E_2^{(2)};$$

$$E_3 = E_3^{(0)} + 2 z E_3^{(1)} \qquad (11)$$

式中各量很容易得出, 此处略。

将上述应变和电场强度表达式代入方程(7)和方程(8), 作分部积分和沿厚度方向积分, 并计及变分的任意性, 就可得不计体力的压电薄壳结构的力平衡表达式和电荷平衡方程及其对应的自然边界条件, 它们分别为

$$\delta u_1^{(0)}: \frac{\partial(A_2 e_{11}^{(0)})}{\partial T_1} + \frac{\partial(A_1 e_{12}^{(0)})}{\partial T_2} + e_{12}^{(0)} \frac{\partial A_1}{\partial T_2} - e_{22}^{(0)} \frac{\partial A_2}{\partial T_1} + \frac{A_1 A_2}{R_1} e_{31}^{(0)} + A_1 A_2 (T_1^+ + T_1^-) = A_1 A_2 (d^{(0)} \ddot{u}_1^{(0)} + d^{(1)} \ddot{u}_1^{(1)})$$

$$e_{11}^{(0)} n_1 + e_{12}^{(0)} n_2 = T_1^{(0)}$$

$$\delta u_1^{(1)}: \frac{\partial(A_2 e_{11}^{(1)})}{\partial T_1} + \frac{\partial(A_1 e_{12}^{(1)})}{\partial T_2} + e_{12}^{(1)} \frac{\partial A_1}{\partial T_2} - e_{22}^{(1)} \frac{\partial A_2}{\partial T_1} + \frac{A_1 A_2}{R_1} e_{31}^{(1)} - A_1 A_2 e_{31}^{(0)} + A_1 A_2 (h T_1^+ - h T_1^-) = A_1 A_2 (d^{(1)} \ddot{u}_1^{(0)} + d^{(2)} \ddot{u}_1^{(1)})$$

$$e_{11}^{(1)} n_1 + e_{12}^{(1)} n_2 = T_1^{(1)}$$

$$\delta u_2^{(0)}: \frac{\partial(A_2 e_{12}^{(0)})}{\partial T_1} + \frac{\partial(A_1 e_{22}^{(0)})}{\partial T_2} + e_{12}^{(0)} \frac{\partial A_2}{\partial T_1} - e_{11}^{(0)} \frac{\partial A_1}{\partial T_2} + \frac{A_1 A_2}{R_2} e_{23}^{(0)} + A_1 A_2 (T_2^+ + T_2^-) = A_1 A_2 (d^{(0)} \ddot{u}_2^{(0)} + d^{(1)} \ddot{u}_2^{(1)})$$

$$e_{12}^{(0)} n_1 + e_{22}^{(0)} n_2 = T_2^{(0)}$$

$$\delta u_2^{(1)}: \frac{\partial(A_2 e_{12}^{(1)})}{\partial T_1} + \frac{\partial(A_1 e_{22}^{(1)})}{\partial T_2} + e_{12}^{(1)} \frac{\partial A_2}{\partial T_1} - e_{11}^{(1)} \frac{\partial A_1}{\partial T_2} + \frac{A_1 A_2}{R_2} e_{23}^{(1)} - A_1 A_2 e_{23}^{(0)} + A_1 A_2 (h T_2^+ - h T_2^-) = A_1 A_2 (d^{(1)} \ddot{u}_2^{(0)} + d^{(2)} \ddot{u}_2^{(1)})$$

$$e_{12}^{(1)} n_1 + e_{22}^{(1)} n_2 = T_2^{(1)}$$

$$\delta u_3^{(0)}: \frac{\partial(A_2 e_{13}^{(0)})}{\partial T_1} + \frac{\partial(A_1 e_{23}^{(0)})}{\partial T_2} - \frac{A_1 A_2}{R_1} e_{11}^{(0)} - \frac{A_1 A_2}{R_2} e_{22}^{(0)} + A_1 A_2 (T_3^+ + T_3^-) = A_1 A_2 (d^{(0)} \ddot{u}_3^{(0)} + d^{(1)} \ddot{u}_3^{(1)}) \qquad (12)$$

$$e_{31}^{(0)} n_1 + e_{23}^{(0)} n_2 = T_3^{(0)}$$

$$\delta u_3^{(1)}: \frac{\partial(A_2 e_{13}^{(1)})}{\partial T_1} + \frac{\partial(A_1 e_{23}^{(1)})}{\partial T_2} - \frac{A_1 A_2}{R_1} e_{11}^{(1)} - \frac{A_1 A_2}{R_2} e_{22}^{(1)} - A_1 A_2 e_{33}^{(0)} + A_1 A_2 (h T_3^+ - h T_3^-) = A_1 A_2 (d^{(1)} \ddot{u}_3^{(0)} + d^{(2)} \ddot{u}_3^{(1)})$$

$$e_{31}^{(1)} n_1 + e_{23}^{(1)} n_2 = T_3^{(1)}$$

$$\delta h^{(0)}: \frac{\partial(A_2 D_1^{(0)})}{\partial T_1} + \frac{\partial(A_1 D_2^{(0)})}{\partial T_2} + A_1 A_2 (w^+ + w^-) = 0$$

$$D_1^{(0)} n_1 + D_2^{(0)} n_2 = w^{(0)}$$

$$\delta h^{(1)}: \frac{\partial(A_2 D_1^{(1)})}{\partial T_1} + \frac{\partial(A_1 D_2^{(1)})}{\partial T_2} - A_1 A_2 D_3^{(0)} + A_1 A_2 (h w^+ - h w^-) = 0 \quad (13)$$

$$D_1^{(1)} n_1 + D_2^{(1)} n_2 = w^{(1)}$$

$$\delta h^{(2)}: \frac{\partial(A_2 D_1^{(2)})}{\partial T_1} + \frac{\partial(A_1 D_2^{(2)})}{\partial T_2} - 2A_1 A_2 D_3^{(1)} + A_1 A_2 (h^2 w^+ + h^2 w^-) = 0$$

$$D_1^{(2)} n_1 + D_2^{(2)} n_2 = w^{(2)}$$

由于假定 $h \ll R_1$, $h \ll R_2$, 在推导(14)式和(15)式中各式时采用了以下近似关系

$$dV = A_1 A_2 dT_1 dT_2 dz;$$

$$1 + z/R_1 \approx 1;\ 1 + z/R_2 \approx 1 \quad (14)$$

式中: 面积分 dS 是在参考面上进行的, 线积分 dc 是沿参考面边界进行的。而 $e_{kl}^{(0)}$, $e_{kl}^{(1)}$ 分别是合力和合力矩; $T_i^{\pm}$ 分别是作用在上下表面的力载荷; $T_i^{(m)}$ 是作用在薄壳周边的力或力矩; $d^{(k)}$ 是惯性矩。定义为

$$e_{kl}^{(m)} = \int z^m e_{kl} dz,\ (m = 0, 1)$$

$$T_i^{(m)} = \int T_i z^m dz,\ (m = 0, 1)$$

$$d^{(k)} = \int \rho z^k dz,\ (k = 0, 1, 2) \quad (15)$$

式中: $w^{\pm}$ 是作用在上下表面的外加电荷面积密度; $D_i^{(m)}$ 是电位移的"m 阶矩"; $w^{(m)}$ 是外加电荷的"m 阶矩"。定义为

$$D_i^{(m)} = \int z^m D_i dz,\ (m = 0, 1, 2)$$

$$w^{(m)} = \int z^m w dz,\ (m = 0, 1, 2) \quad (16)$$

需说明的是, (13)式仅适合以下两种情况:

(1) 压电片作为传感元件, 此时在其上下表面无外加自由电荷, 即 $D_3|_{\pm h} = 0$。

(2) 压电片作为作动元件, 此时驱动信号仅仅是在上下表面外加电荷, 即 $D_3|_{\pm h} = D_3^{\pm}$。

而当压电元作为作动器, 且驱动信号是在上下表面作用电势差时(这是更为常用的一种作动方式), 其电荷平衡方程具有与(13)式不同的形式。此时, 可设其上下表面的作用电势分别为 h^+ 和 h^-, 即

$$h|_{h} = h^+;\ h|_{-h} = h^- \quad (17)$$

将(9)式中第四式代入上式可得

$$h^{(1)} = (h^+ - h^-)/2h;$$

$$h^{(2)} = -h^{(0)}/h^2 + (h^+ + h^-)/2h^2 \quad (18)$$

综合(8)式、(11)式和(18)式, 就可以得到压电薄壳结构作为驱动元件且驱动信号为电压时的电荷平衡方程及其边界条件

$$\delta h^{(0)}: \frac{\partial[A_2(D_1^{(0)} - D_1^{(2)}/h^2)]}{\partial T_1} + \frac{\partial[A_1(D_2^{(0)} - D_2^{(2)}/h^2)]}{\partial T_2} - \frac{2A_1 A_2 D_3^{(1)}}{h^2} = 0$$

$$(D_1^{(0)} - D_1^{(2)}/h^2) n_1 + (D_2^{(0)} - D_2^{(2)}/h^2) n_2 = w^{(0)} - w^{(2)}/h^2 \quad (19)$$

当电学边界条件为(17)式所示时, (19)式将替代(13)式为电荷平衡方程式和边界条件。

将本构方程(1)积分, 得压电梯度壳的本构关系为

$$\begin{matrix} e^{(0)} \\ e^{(1)} \end{matrix} = \begin{matrix} A & B \\ B & D \end{matrix} \begin{matrix} X^{(0)} \\ X^{(1)} \end{matrix} - \begin{matrix} P & Q & R \\ Q & R & H \end{matrix} \begin{matrix} E^{(0)} \\ E^{(1)} \\ E^{(2)} \end{matrix} \quad (20)$$

$$\begin{matrix} D^{(0)} \\ D^{(1)} \\ D^{(2)} \end{matrix} = \begin{matrix} P^T & Q^T \\ Q^T & R^T \\ R^T & H^T \end{matrix} \begin{matrix} X^{(0)} \\ X^{(1)} \end{matrix} + \begin{matrix} K & L & M \\ L & M & N \\ M & N & O \end{matrix} \begin{matrix} E^{(0)} \\ E^{(1)} \\ E^{(2)} \end{matrix} \quad (21)$$

式中:

$$e^{(n)} = \{e_{11}^{(n)}, e_{22}^{(n)}, e_{33}^{(n)}, e_{23}^{(n)}, e_{31}^{(n)}, e_{12}^{(n)}\}^T$$

$$X^{(n)} = \{X_{11}^{(n)}, X_{22}^{(n)}, X_{33}^{(n)}, X_{23}^{(n)}, X_{31}^{(n)}, X_{12}^{(n)}\}^T \quad (n = 0, 1)$$

$$D^{(n)} = \{D_1^{(n)}, D_2^{(n)}, D_3^{(n)}\}^T$$

$$E^{(n)} = \{E_1^{(n)}, E_2^{(n)}, E_3^{(n)}\}^T \quad (n = 0, 1, 2) \quad (22)$$

在上式中需考虑横向剪切的修正因子(本文中取为 5/6), 矩阵的元素定义为

$$\{A_{ij}, B_{ij}, D_{ij}\} = \int_{-h}^{h} C_{ij}(1, z, z^2) dz;$$

$$\{P_{ij}, Q_{ij}, R_{ij}, H_{ij}\} = \int_{-h}^{h} e_{ij}(1, z, z^2, z^3) dz;$$

$$\{K_{ij}, L_{ij}, M_{ij}, N_{ij}, O_{ij}\} = \int_{-h}^{h} X_{ij}(1, z, z^2, z^3, z^4) dz \quad (23)$$

将式(20)和式(21)代入力、电平衡方程式(12)和式(13), 得到用 $u_1^{(0)}$, $u_1^{(1)}$, …, $h^{(2)}$ 等表示的控制微分方程为

$$L_d U + L_c h + P = L_d U$$

$$L_c^* U + L_w h + Q = 0 \quad (24)$$

式中:

$$U = \{u_1^{(0)}, u_1^{(1)}, u_2^{(0)}, u_2^{(1)}, u_3^{(0)}, u_3^{(1)}\}^T$$

$$h = \{h^{(0)}, h^{(1)}, h^{(2)}\}^T \quad (25)$$

矩阵 $L_d = L_{d1}\frac{\partial^2}{\partial T_1^2} + L_{d2}\frac{\partial^2}{\partial T_2^2} + L_{d3}\frac{\partial^2}{\partial T_1 \partial T_2} + L_{d4}\frac{\partial}{\partial T_1} + L_{d5}\frac{\partial}{\partial T_2} + L_{d6}$。

矩阵 L_c 和 L_w 可通过将上式中的 L_{di} 分别用 L_{ci} 和 L_{wi} 代替而得到。

矩阵 $L_c^* = L_{c1}^T \frac{\partial^2}{\partial T_1^2} + L_{c2}^T \frac{\partial^2}{\partial T_2^2} + L_{c3}^T \frac{\partial^2}{\partial T_1 \partial T_2} - L_{c4}^T \frac{\partial}{\partial T_1} - L_{c5}^T \frac{\partial}{\partial T_2} + L_{c6}^T$。

$$P = \{P_1^{(0)}, P_1^{(1)}, P_2^{(0)}, P_2^{(1)}, P_3^{(0)}, P_3^{(1)}\}$$

$$Q = \{q^{(0)}, q^{(1)}, q^{(2)}\}$$

$$P_i^{(0)} = A_1 A_2 (T_i^+ + T_i^-);$$

$$P_i^{(1)} = A_1 A_2 (hT_i^+ - hT_i^-) \quad (i = 1, 2, 3)$$

$$q^{(n)} = A_1 A_2 (h^{(n)} w^+ + (-h)^{(n)} w^-) \quad (n = 0, 1, 2) \tag{26}$$

而压电元作为作动元且驱动信号是电压时方程也就不难获得(将本构方程代入(14)式和(21)式即可,此处略)。

2　四边简支的压电薄壳

本节具体讨论了在上下表面承受按正弦或余弦函数变化的力、电载荷的四边简支压电浅壳的静态特性(此时,可认为拉梅系数和曲率半径均为常数)。不难验证,当位移、电势具有如下分布规律时,所有简支边界都得到满足。

$$\{u_1^{(0)}, u_1^{(1)}\} = \{U^{(0)}, U^{(1)}\} \cos aT_1 \sin bT_2$$

$$\{u_2^{(0)}, u_2^{(1)}\} = \{V^{(0)}, V^{(1)}\} \sin aT_1 \cos bT_2$$

$$\{u_3^{(0)}, u_3^{(1)}\} = \{W^{(0)}, W^{(1)}\} \sin aT_1 \sin bT_2 \tag{27}$$

$$\{h^{(0)}, h^{(1)}, h^{(2)}\} = \{J^{(0)}, J^{(1)}, J^{(2)}\} \sin aT_1 \sin bT_2 \tag{28}$$

其中: $a = m\pi/L_1$, $b = n\pi/L_2$, m 和 n 是整数, L_1 和 L_2 是壳在 T_1 和 T_2 方向上的长度; $U^{(0)}, V^{(0)}, W^{(0)}, J^{(0)}$ 等是要求解的未知数。

当压电元作为传感元或作动元但驱动信号是自由面电荷密度时,设只在壳体上下表面外加正弦变化的静态力、电荷载(如果不是需按 Fourier 级数展开)为 $T_3^{\pm} \sin aT_1 \sin bT_2$ 和 $w^{\pm} \sin aT_1 \sin bT_2$,代(27)式和(28)式入相应的力、电平衡控制微分方程,则得到该问题的代数方程

$$[6]\{x\} = \{F\} \tag{29}$$

其中: $\{x\}$ 是未知的位移和电势向量; $\{F\}$ 是外加载荷; $[6]$ 是 9×9 对称系数阵。

$$x = \{U^{(0)}, U^{(1)}, V^{(0)}, V^{(1)}, W^{(0)}, W^{(1)}, J^{(0)}, J^{(1)}, J^{(2)}\}^T$$

$$F = \{0 \quad 0 \quad 0 \quad 0 \quad -A_1A_2(T^+ + T^-) - A_1A_2h(T^+ - T^-) - A_1A_2(w^+ + w^-) - A_1A_2h(w^+ - w^-) - A_1A_2h^2(w^+ + w^-)\}^T \tag{30}$$

当压电元作为作动元且驱动信号是电压时,壳体上下表面外加正弦变化的静态力电荷载为 $T_3^{\pm} \sin aT_1 \sin bT_2$ 和 $h^{\pm} \sin aT_1 \sin bT_2$,相应的代数方程是

$$[6]\{x\} = \{F\} \tag{31}$$

而 $[6]$ 是 7×7 的对称系数阵,它与 $[6]$ 的关系为

$$6_{ij} = 6_{ij}, \quad i \le 6, j \le 6;$$

$$6_{7j} = 6_{7j} - 6_{9j}/h^2, \quad j \le 6;$$

$$6_{77} = 6_{77} - 6_{99}/h^2 \tag{32}$$

求解代数方程(29)或方程(31)就可以得到简支压电梯度薄壳在不同载荷下的力电学特性。

3　数值结果

由于至今尚未见有压电梯度壳的精确分析的发表,为了验证本文中的高阶理论,这里首先分析了一种特殊情况,即非均匀简支压电层合板,并与精确结果作了比较。最后求解了简支压电梯度圆柱壳的静态特性。

3.1　非均匀简支压电层合板

Cheng 等[11]用渐进展开法精确地分析了一三层(0°/90°/0°)PVDF 压电板,每层都是均匀的且具有相同厚度,板宽趋于无限大,即 $a/b \to 0$。在其上表面作用了正弦分布的法向力或电势,取 $m = n = 1$。PVDF 的材料常数为

$$C_{11} = 238.24, \ C_{22} = 23.6, \ C_{12} = 3.98,$$

$$C_{33} = 10.64, \ C_{13} = 2.19, \ C_{23} = 1.92, \ C_{44} = 4.4,$$

$$C_{55} = 2.15, \ C_{66} = 6.43 (\mathrm{GPa}),$$

$$e_{31} = -0.13, \ e_{32} = -0.145, \ e_{33} = -0.276,$$

$$e_{15} = 0.135, \ e_{24} = -0.009 (\mathrm{Cm^{-2}}),$$

$$\in_{11} = 1.1068, \ \in_{22} = \in_{33} = 1.0607 (\mathrm{CVm^{-1}})$$

便于比较,力学参量和电学参量无量纲化 $u_i = u_i/Pa$, $e_{ij} = e_{ij}/Pc^*$, $h = e^* h/Pac^*$。其中: a 是板的长度; 外加力载荷时 $P = -T_3^+/c^*$,外加电载荷时 $P = h^+(e^*/ac^*)$; $c^* = 1\mathrm{N/m^2}$; $e^* = 1\mathrm{C/m^2}$。如果前一节的公式中,取拉梅系数、主曲率半径分别为 $A_1 = A_2 = 1$, $R_1 = R_2 = \infty$,材料性能参数为 z 的分段常函数,那么所得结果即为非均匀压电层合板的高阶理论解。

图 2～图 3 所示为法向正弦变化的力载荷作用下,电势 h 和面内应力 e_{11} 在不同的宽厚比时沿厚度方向的分布情况,实线为高阶理论的结果,虚线为三

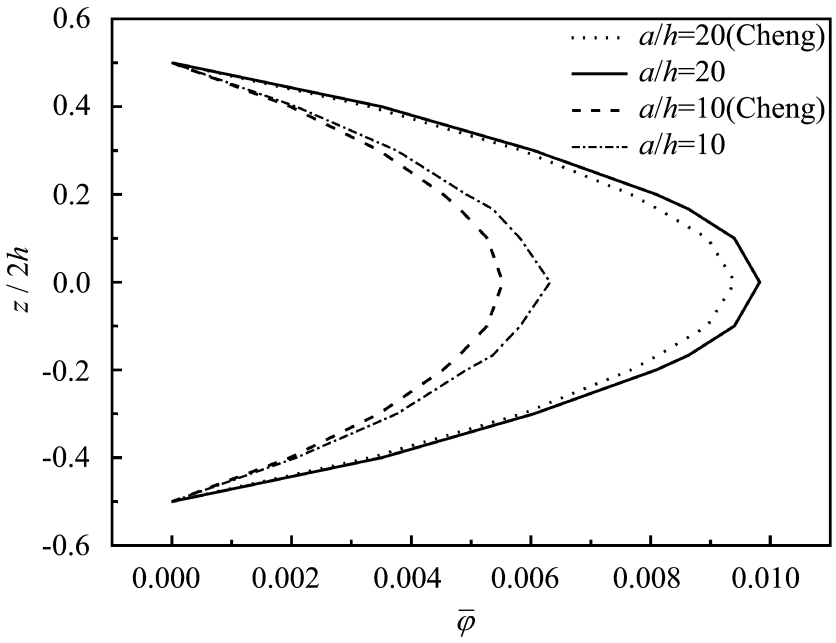

图 2 压电层合板受法向力荷载的电势分布图

Fig. 2 Distribution of electric potential for a three-layer PVDF laminate under the mechanical load

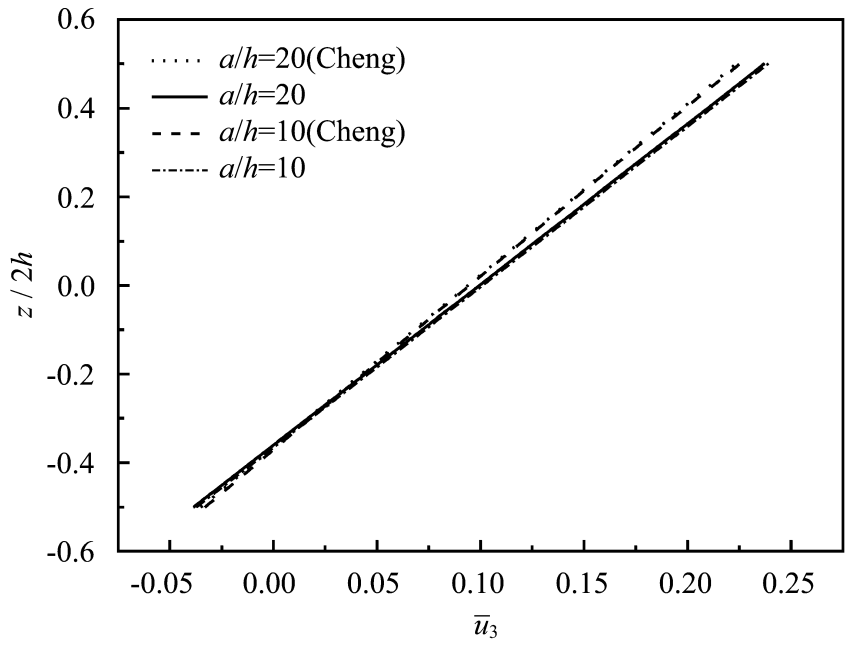

图 4 压电层合板受法向电荷载的横向位移分布图

Fig. 4 Distribution of out-of-plane displacement for a three-layer PVDF laminate under the electric load

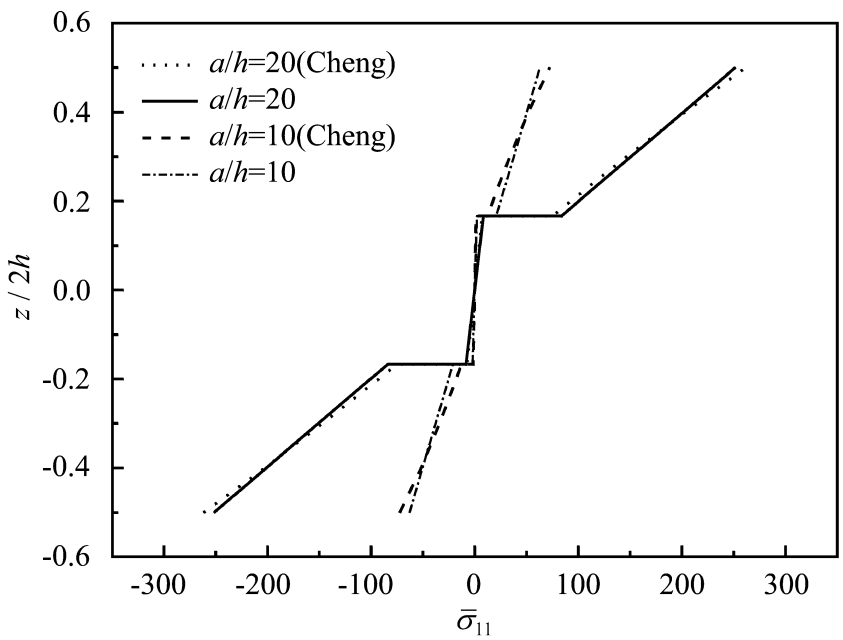

图 3 压电层合板受法向力荷载的应力分布图

Fig. 3 Distribution of inplane stress for a three-layer PVDF laminate under the mechanical load

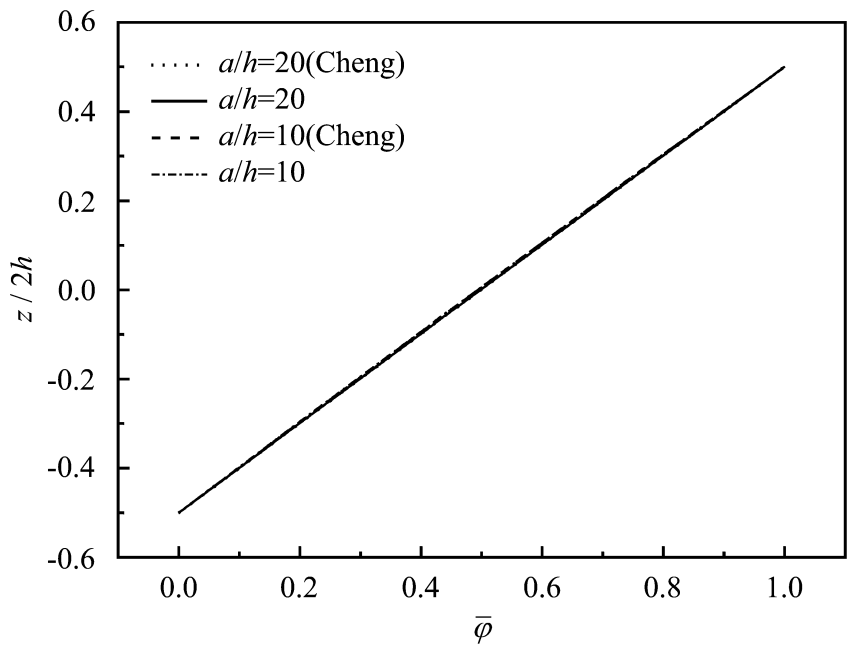

图 5 压电层合板受法向电荷载的电势分布图

Fig. 5 Distribution of electric potential for a three-layer PVDF laminate under the mechanical load

维精确解的结果(以下均同)。图 4～图 5 所示为法向正弦变化的电势作用下, 法向位移 u_3 和电势 h 在不同的宽厚比时沿厚度方向的分布情况。从图中可以看出: 高阶理论中的横向位移和二次电势分布的假设是必要的。由高阶理论得出的非均匀压电层合板解与三维精确解吻合得很好, 且板越薄则两者的结果越一致。另外, 面内应力 e_{11} 沿层间不连续。

3.2 简支压电梯度圆柱壳

这里, 用前述的高阶理论分析一上表面受正弦变化的横向力荷载, 作为传感器或作动器的简支压电梯度圆柱壳, 并与均匀圆柱壳比较。作为特例, 只有压电系数沿厚度方向变化, 其余参数均为常量。本例中, 压电系数沿厚度方向变化规律为 $e(z) = e^0(1 + z/H)$ 和 $e(z) = 3e^0(1/2 + z/H)^2$。其中: $H = 2h$ 是壳的厚度; z 的变化范围从 $-h$ 到 h。材料为横观各向同性压电陶瓷 PZT-4, 性能参数由 Beilincourt[12] 给出。

$$C_{11} = 13.9;\ C_{12} = 7.78;\ C_{13} = 7.43;$$
$$C_{33} = 11.5;\ C_{44} = 2.56(10^{10}\,\mathrm{Pa});$$
$$e_{31} = -5.2;\ e_{33} = 15.1;\ e_{15} = 12.7(\mathrm{Cm}^{-2});$$
$$\in_{11} = 6.46;\ \in_{22} = 5.62(\mathrm{CVm}^{-1})$$

此时, 在高阶理论中取 $A_1 = A_2 = 1$; $R_1 = R_0$; $R_2 = \infty$; 其中 R_0 为壳体中面半径。在数值计算中取 $R_0 = 1\mathrm{m}$; $L_1 = \pi/3$; $L_2 = 4\mathrm{m}$; $h = 0.1\mathrm{m}$。解的无量纲化为 $u_3 = u_3 10^8/2h$; $e_{11} = e_{11}/c^*$。当施加的是力荷载时, $h = h e^*/c^* h$。其中: $c^* = 1\mathrm{N/m}^2$; $e^* = 1\mathrm{C/m}^2$。当施加的是电势时, $h = h/V_0$。

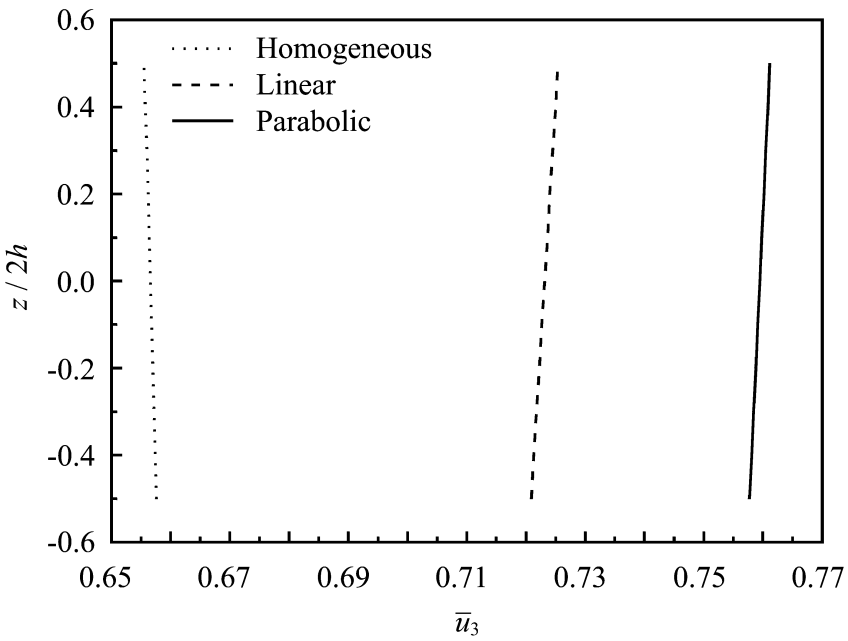

图 6　压电壳受单位法向力荷载的横向位移分布图

Fig. 6　Distribution of out-of-plane displacement for piezoelectric shells under unit mechanical load

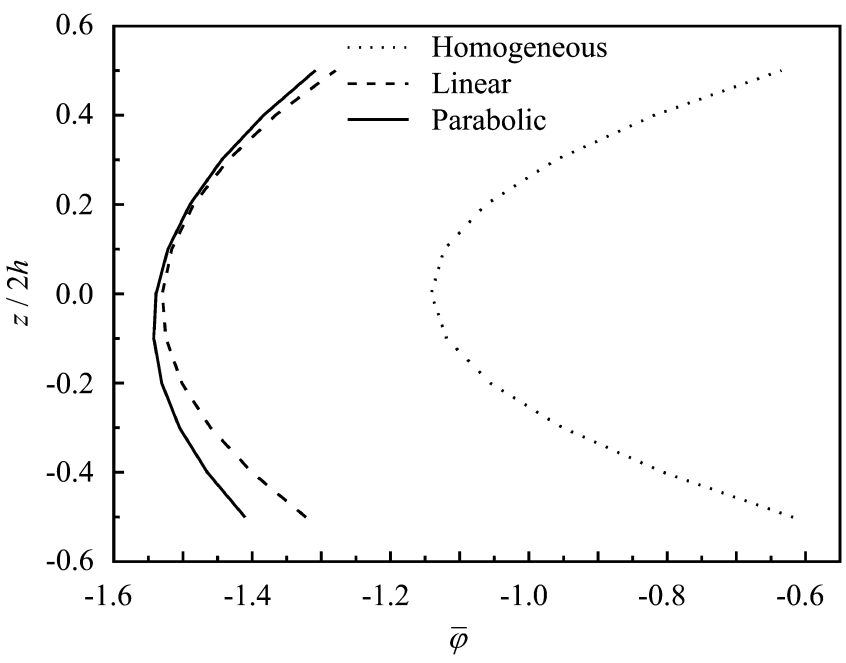

图 7　压电壳受单位法向力荷载的电势分布图

Fig. 7　Distribution of electric potential for piezoelectric shells under unit mechanical load

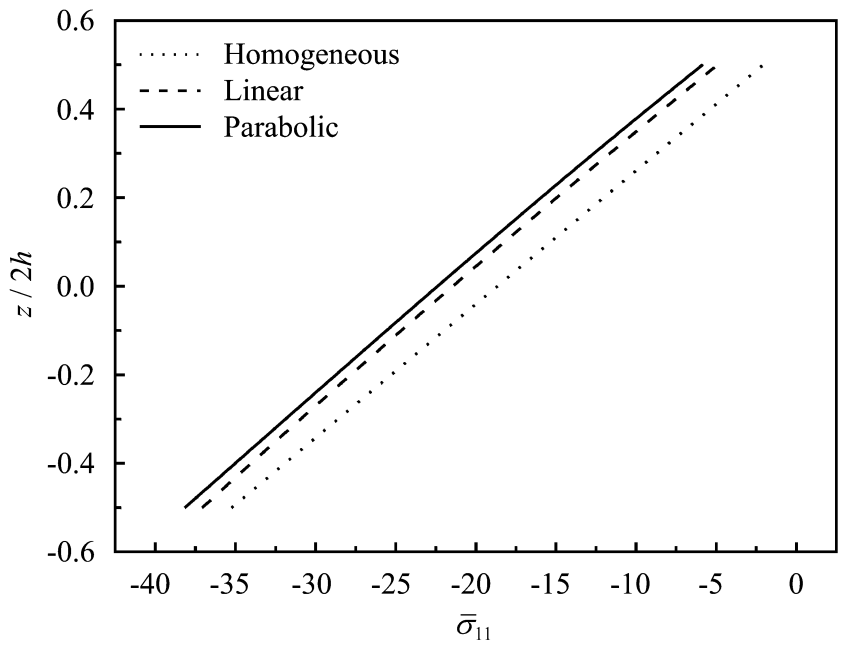

图 8　压电壳受单位法向力荷载的应力分布图

Fig. 8　Distribution of inplane stress for piezoelectric shells under unit mechanical load

图 6～图 8 为受单位法向力荷载，上下表面电位移为零时，梯度壳与均匀壳的径向位移 u_3，电势 h 和环向应力 e_{11} 沿厚度方向的分布规律(实线为二次梯度解，虚线为线性梯度解，点划线为均匀解，以下相同)。从图中可发现：梯度壳的径向位移 u_3 和电势 h 都显著大于均匀壳的，且二次梯度壳的最大，同时这三种壳的电势都是二次分布的，而梯度壳的应力 e_{11} 仅略大于均匀壳的。图 9～图 11 为作用单位正弦变化的电势时，梯度壳与均匀壳的径向位移 u_3，电势 h 和环向应力 e_{11} 沿厚度方向的分布规律。从图中可看出：梯度壳的径向位移 u_3 和电势 h 同样都大于均匀壳的，而二次梯度壳的应力 e_{11} 则小于均匀壳的，但线性梯度壳的略大于均匀壳的。壳的径向位移

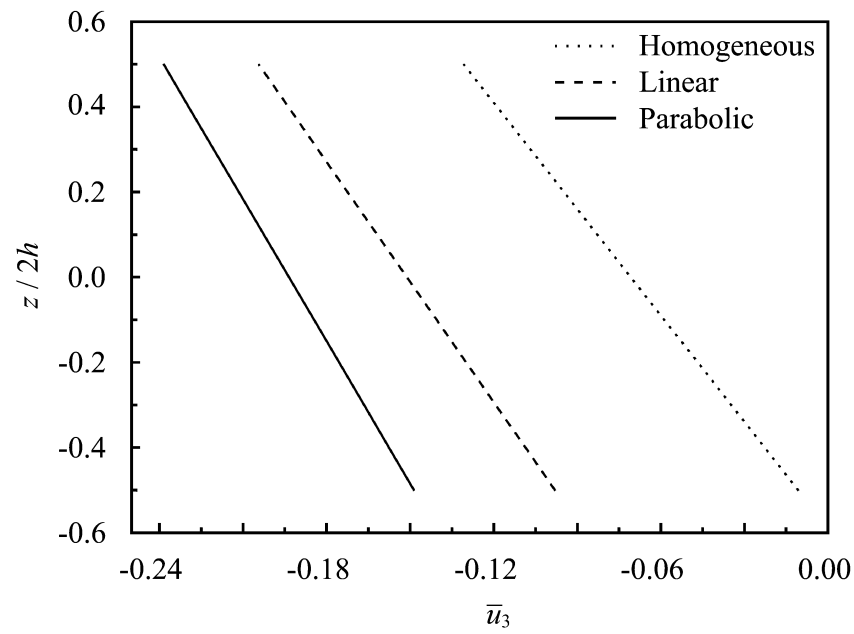

图 9　压电壳作用单位正弦电压的横向位移分布图

Fig. 9　Distribution of out-of-plane displacement for piezoelectric shells under unit sinusoid voltage

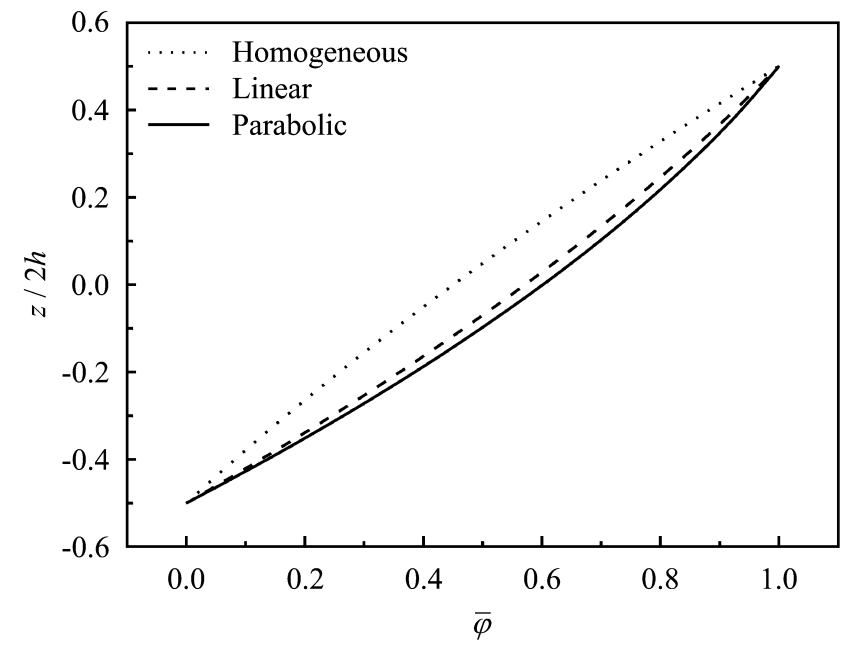

图 10　压电壳作用单位正弦电压的电势分布图

Fig. 10　Distribution of electric potential for piezoelectric shells under unit sinusoid voltage

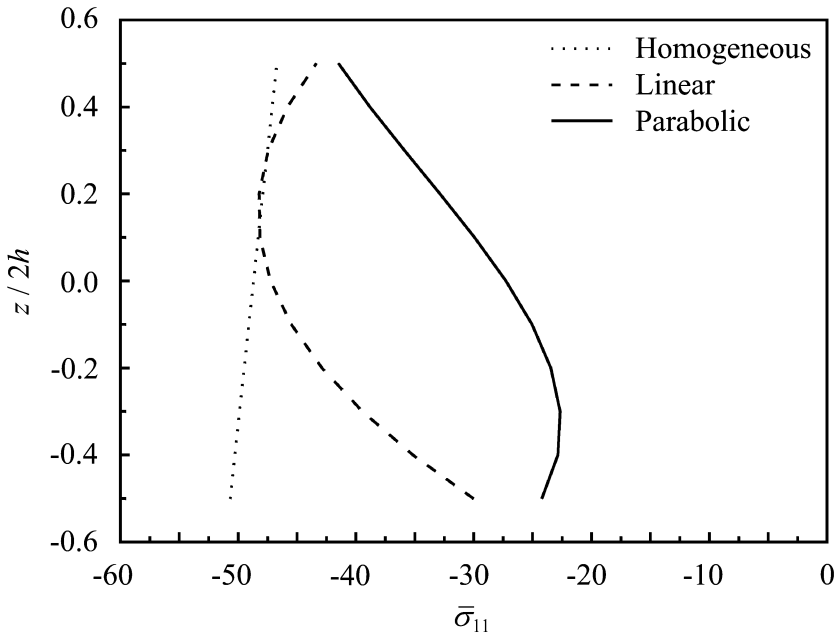

图 11 压电壳作用单位正弦电压的应力分布图

Fig. 11 Distribution of inplane stress for piezoelectric shells under unit sinusoid voltage

沿厚度方向线性变化而不是常量,因此高阶理论中的横向位移分布的假设是必要的。但此时,壳的电势分布接近线性。图中均匀壳体的数值与三维精确理论计算结果[9]相符合。从以上图中可得出:梯度壳的性能整体上要优于均匀壳,对压电材料系数的梯度变化规律进行优化可得到期望的优越性能。

4 结 论

本文作者提出了一个简单而有效的求解压电梯度薄壳的力、电行为特性的高阶理论。所得结果可退化至梁、板及圆柱壳等多种特殊情况。根据所得方程讨论了一非均匀压电层合板实例,并与三维精确结果作了对比,两者很好地吻合,表明该理论的正确性。最后具体分析了压电梯度圆柱壳的力、电响应特性,给出了其位移、应力、电势沿厚度方向的变化规律,并与均匀压电圆柱壳相比较,显示了材料系数梯度变化对响应的影响。

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