第3卷 第1期 复 合 材 料 学 报 Vol. 3 No. 1

1986年3月 ACTA MATERIAE COMPOSITAE SINICA March 1986

不对称正交层板室温形状的简单计算方法

陈烈民

（中国空間技术研究院，北京5106信箱）

摘 要

本文采用了广义变分原理和大挠度薄板理論导出了計算不对称正交层板室温形状的簡单方法，它的計算結果与以前通过复杂計算所得的結果非常一致，但本法更适于工程上的实际应用。

一、引 言

不对称层板在高温固化后的室温形状分析是一个对复合材料设计和制造具有实际意义的问题。*Hyer*[1][2][3]曾用大挠度薄板理论和最小势能原理计算了不对称正交矩形层板的室温形状，但要进行复杂的高次代数方程组求数值解，在工程上应用困难。

本文采用大挠度薄板理论和广义变分原理，首先导出了以中面应力函数和挠度函数为变量的泛函形式。然后用变分法导出了计算层板室温形状的代数方程，由于方程的形式很简单，适于工程实际应用。

二、广义变分原理中泛函形式的确定

考虑边长为$2a$、$2b$，厚度为h的不对称铺层的正交矩形板。板的中面位移为u_0、v_0，挠度为w，中面应变为ε_{x0}、ε_{y0}、γ_{xy0}，曲率变化为κ_x、κ_y、κ_{xy}，中面内力为N_x、N_y、N_{xy}，内力矩为M_x、M_y、M_{xy}，则受均匀温度（$-\Delta t$）时板的本构关系为：

本文1985年8月22日收到

$$\left\{\begin{array}{c}N_x\\N_y\\N_{xy}\\\hline M_x\\M_y\\M_{xy}\end{array}\right\}=\left[\begin{array}{ccc|ccc}A_{11}&A_{12}&0&B_{11}&B_{22}&0\\A_{12}&A_{22}&0&B_{12}&B_{22}&0\\0&0&A_{66}&0&0&B_{66}\\\hline B_{11}&B_{12}&0&D_{11}&D_{12}&0\\B_{12}&B_{22}&0&D_{12}&D_{22}&0\\0&0&B_{66}&0&0&D_{66}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{x0}\\\varepsilon_{y0}\\\gamma_{xy0}\\\hline \kappa_x\\\kappa_y\\\kappa_{xy}\end{array}\right\}+\left\{\begin{array}{c}n_1\\n_2\\0\\\hline m_1\\m_2\\0\end{array}\right\}(\Delta t)\qquad(1)$$

或表示为缩写矩阵形式为：

$$\left\{\begin{array}{c}\mathbf{N}\\\hline\mathbf{M}\end{array}\right\}=\left\{\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\\hline\mathbf{B}&\mathbf{D}\end{array}\right\}\left\{\begin{array}{c}\varepsilon_0\\\hline\kappa\end{array}\right\}+\left\{\begin{array}{c}\mathbf{n}\\\hline\mathbf{m}\end{array}\right\}(\Delta t)\qquad(1)$$

其中

$$A_{ij}=\int_{-h/2}^{h/2}Q_{ij}dz,\quad B_{ij}=\int_{-h/2}^{h/2}Q_{ij}zdz,$$

$$D_{ij}=\int_{-h/2}^{h/2}Q_{ij}z^2dz,\quad (i,\ j=1,\ 2;\ 2,\ 2;\ 6,\ 6)$$

$$n_1=\int_{-h/2}^{h/2}(\alpha_1Q_{11}+\alpha_2Q_{12})dz,\quad n_2=\int_{-h/2}^{h/2}(\alpha_1Q_{12}+\alpha_2Q_{22})dz$$

$$m_1=\int_{-h/2}^{h/2}(\alpha_1Q_{11}+\alpha_2Q_{12})zdz,\quad m_2=\int_{-h/2}^{h/2}(\alpha_1Q_{12}+\alpha_2Q_{22})zdz$$

这里1，2是表示 x，y 方向上参数的脚注，Q_{ij} 为各层材料的缩减弹性系数，α_1，α_2为各层材料沿 x、y 方向的温度膨胀系数。（Δt）可理解为复合材料固化温度与室温之差。

现将广义变分原理[4]推广到本文考虑大挠度和温度变化的情形，其泛函形式为：

$$\begin{aligned}\Pi_3=\int_{-a}^{a}\int_{-b}^{b}\Big\{&\frac{1}{2}\Big[\boldsymbol{\varepsilon}_0^T\mathbf{A}\boldsymbol{\varepsilon}_0+2\boldsymbol{\varepsilon}_0^T\mathbf{B}\mathbf{K}+\mathbf{K}^T\mathbf{D}\mathbf{K}+2\mathbf{n}^T\boldsymbol{\varepsilon}_0\Delta t\\&+2\mathbf{m}^T\mathbf{K}\Delta t\Big]-\Big[\varepsilon_{x0}-\frac{\partial u_0}{\partial x}-\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big]N_x\\&-\Big[\varepsilon_{y0}-\frac{\partial v_0}{\partial y}-\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big]N_y\\&-\Big[\gamma_{xy0}-\frac{\partial u_0}{\partial y}-\frac{\partial v_0}{\partial x}-\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big]N_{xy}\\&-\Big(\kappa_x+\frac{\partial^2 w}{\partial x^2}\Big)M_x-\Big(\kappa_y+\frac{\partial^2 w}{\partial y^2}\Big)M_y\\&-\Big(\kappa_{xy}+2\frac{\partial^2 w}{\partial xzy}\Big)M_{xy}\Big\}dxdy\end{aligned}\qquad(2)$$

利用（1）式及分部积分，（2）可改写成：

$$\Pi_3=\int_{-a}^{a}\int_{-b}^{b}\Big\{\frac{1}{2}\Big[\mathbf{N}^T\mathbf{A}^{-1}\mathbf{N}+2\mathbf{K}^T\mathbf{B}\mathbf{A}^{-1}\mathbf{N}+\mathbf{K}^T(\mathbf{D}-\mathbf{B}\mathbf{A}^{-1}\mathbf{B})\mathbf{K}$$

$$-2(\mathbf{K}^T\mathbf{B}\mathbf{A}^{-1}-\mathbf{N}^T\mathbf{A}^{-1})\mathbf{m}\Delta t+2\mathbf{K}_T\mathbf{m}\Delta t\Big]-\Big(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\Big)u_0-\Big(\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}\Big)v_0+\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2N_x+\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2N_y+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}N_{xy}-\Big(\kappa_x+\frac{\partial^2 w}{\partial x^2}\Big)M_x-\Big(\kappa_y+\frac{\partial^2 w}{\partial y^2}\Big)M_y-\Big(\kappa_{xy}+2\frac{\partial^2 w}{\partial x\partial y}\Big)M_{xy}\Big\}dxdy+\int_{-b}^{b}(N_xu_0+N_{xy}v_0)_{x=\pm a}dy+\int_{-a}^{a}(N_{xy}u_0+N_yv_0)_{y=\pm b}dx \quad (3)$$

假设中面内力满足所有边界条件即 $x=\pm a$：$N_x=N_{xy}=0$，$y=\pm b$：$N_y=N_{xy}=0$，且又满足平衡条件，即存在中面应力函数ϕ：

$$N_x=\frac{\partial^2\phi}{\partial y^2},\qquad N_y=\frac{\partial^2\phi}{\partial x^2},\qquad N_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \quad (4)$$

同时假设满足如下几何条件：

$$\kappa_x=-\frac{\partial^2 w}{\partial x^2},\qquad \kappa_y=-\frac{\partial^2 w}{\partial y^2},\qquad \kappa_{xy}=-2\frac{\partial^2 w}{\partial x\partial y} \quad (5)$$

利用(1)、(4)、(5)，(3)可改写为：

$$\Pi_3=\frac{1}{2}\int_{-a}^{a}\int_{-b}^{b}\Big[-a_{11}\Big(\frac{\partial^2\phi}{\partial y^2}\Big)^2-2a_{12}\Big(\frac{\partial^2\phi}{\partial x^2}\Big)\Big(\frac{\partial^2\phi}{\partial y^2}\Big)-a_{22}\Big(\frac{\partial^2\phi}{\partial x^2}\Big)^2-a_{66}\Big(\frac{\partial^2\phi}{\partial x\partial y}\Big)^2-2(B_{11}a_{11}-B_{12}a_{12})\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-2(B_{11}a_{12}+B_{12}a_{22})\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial x^2}-2(B_{12}a_{11}+B_{22}a_{12})\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}-2(B_{12}a_{12}+B_{22}a_{22})\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}+4B_{66}a_{66}\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\phi}{\partial x\partial y}+d_{11}h^3\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2+2d_{12}h^3\Big(\frac{\partial^2 w}{\partial x^2}\Big)\Big(\frac{\partial^2 w}{\partial y^2}\Big)+d_{22}h^3\Big(\frac{\partial^2 w}{\partial y^2}\Big)^2+4(D_{66}-B_{66}a_{66})\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2+2e_{11}h^2\frac{\partial^2 w}{\partial x^2}\Big]\Delta t+2e_{22}h^2\frac{\partial^2 w}{\partial y^2}\Big]\Delta t-2(a_{11}n_1+a_{12}n_2)\frac{\partial^2\phi}{\partial y^2}\Big]\Delta t+2(a_{12}n_1+a_{22}n_2)\frac{\partial^2\phi}{\partial x^2}\Big]\Delta t+\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2\Big(\frac{\partial^2\phi}{\partial y^2}\Big)$$

$$+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\left(\frac{\partial^2 \phi}{\partial x^2}\right)-2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial^2 \phi}{\partial x \partial y}\right)\Big]dx\,dy \tag{6}$$

其中

$$d_{11}=(D_{11}-a_{11}B_{11}^2-a_{22}B_{12}^2-2a_{12}B_{11}B_{12})/h^3$$
$$d_{22}=(D_{22}-a_{22}B_{22}^2-a_{11}B_{12}^2-2a_{12}B_{12}B_{22})/h^3$$
$$d_{12}=(D_{12}-a_{12}B_{11}B_{22}-a_{11}B_{11}B_{12}-a_{22}B_{12}B_{22})/h^3$$
$$e_{11}=[(B_{11}a_{11}+B_{12}a_{12})n_1+(B_{11}a_{12}+B_{12}a_{22})n_2-m_1]/h^2$$
$$e_{22}=[(B_{12}a_{11}+B_{22}a_{12})n_1+(B_{12}a_{12}+B_{22}a_{22})n_2-m_2]/h^2$$
$$a_{11}=A_{22}/(A_{11}A_{22}-A_{12}^2)$$
$$a_{11}=A_{11}/(A_{11}A_{22}-A_{12}^2)$$
$$a_{12}=-A_{12}/(A_{11}A_{22}-A_{12}^2)$$
$$a_{66}=1/A_{66}$$

三、层板室温形状計算方程的建立

假设：

$$w=-\frac{1}{2h}(K_1x^2-K_2y^2) \tag{7}$$

$$\phi=\phi_1h^3\left(1-\frac{x^2}{a^2}\right)^2\left(1-\frac{y_2}{b^2}\right)^2+\phi_2h^3\left(1-\frac{x^2}{a^2}\right)^4\left(1-\frac{y^2}{b^2}\right)^4 \tag{8}$$

其中ϕ_1、ϕ_2、K_1、K_2为待定系数，K_1/h，K_2/h相当于沿x、y向板的曲率值。上述(7)式采用了与文献[1]相同的形式，而(8)的函数形式可保证中面内力满足所有边界条件。以(7)、(8)代入(6)，积分后得到：

$$\frac{\Pi_3}{4abh}=-\frac{1}{2}C_1\phi_1^2-\frac{1}{2}C_2\phi_2^2-C_3\phi_1\phi_2+\frac{1}{2}d_{11}K_1^2+\frac{1}{2}d_{22}K_2^2$$
$$+d_{12}K_1K_2-e_{11}K_1-e_{22}K_2-C_4K_1K_2-C_5K_1K_2\phi_2 \tag{9}$$

其中

$$C_1=5.20\left(\frac{h}{a}\right)^4(a_{11}\lambda^4+0.571a_{12}\lambda^2+0.286a_{66}\lambda^2+a_{22})h$$

$$C_2=5.88\left(\frac{h}{a}\right)^4(a_{11}\lambda^4+0.720a_{12}\lambda^2+0.360a_{66}\lambda^2+a_{22})h$$

$$C_3=3.33\left(\frac{h}{a}\right)^4(a_{11}\lambda^4+0.840a_{12}\lambda^2+0.120a_{66}\lambda^2+a_{11})h$$

$$C_4=1.284$$
$$C_5=0.165$$
$$\lambda=a/b$$

根据变分原理，$\frac{\partial \Pi_3}{\partial \phi_1}=\frac{\partial \Pi_3}{\partial \phi_2}=\frac{\partial \Pi_3}{\partial K_1}=\frac{\partial \Pi_3}{\partial K_2}=0$，由此可导出含$\phi_1$、$\phi_2$、$K_1$、的四个代数方程，消去$\phi_1$、$\phi_2$后，得出如下两个方程：

$$d_{11}K_1^5-e_{11}\Delta tK_1^4+\frac{2d_{11}d_{22}}{C}K_1^3-\frac{(2e_{11}d_{22}+e_{22}d_{12})\Delta t}{C}K_1^2$$

$$+\left[\frac{(d_{11}d_{22}-d_{12}^2)d_{22}}{C}+\frac{e_{22}^2}{C}-(\Delta t)^2\right]K_1$$

$$-\frac{(d_{22}e_{11}-d_{12}e_{22})d_{22}\Delta t}{C^2}=0 \tag{11}$$

$$K_2=-\frac{d_{12}K_1-e_{22}\Delta t}{d_{22}+CK_1^2} \tag{12}$$

其中 $$C=(C_4^2C_2+C_5^2C_1-2C_3C_4C_5)/(C_1C_2-C_3^2)$$

这里应指出，即使在（7）中取更多项数的多项式，只要仍使满足中面内力边界条件，计算表示，最后仍导出与（11）、(12)相同的方程形式，仅对C值有所改进。因此，(11)、(12）具有较普遍的意义。

由于（11）、（12）不需联好求解，可以很方便地先后求出K_1及K_2值。

对于0°与90°层交替铺设情形，（11)、(12）还可以简化。此时$B_2=-B_{11}$，$B_{12}=0$，$n_1=n_2$，$m_2=-m_1$，因而$d_{22}=d_{11}$，$e_{22}=-e_{11}$，（11）式可分解因式成两个独立方程：

$$K_1^3+\frac{(d_{11}-d_{12})}{C}K_1-\frac{e_{11}}{C}\Delta t=0 \tag{13a}$$

$$d_{11}K_1^2-e_{11}\Delta tK_1+\frac{d_{11}(d_{11}+d_{12})}{C}=0 \tag{13b}$$

(13a)为三次代数方程，可判断出只有一个实根为$K_1^{(1)}$，(13b)为二次代数方程，有两个实根为：

$$\left.\begin{matrix}K_1^{(2)}\\K_1^{(3)}\end{matrix}\right\}=\frac{1}{2}\left\{\frac{e_{11}\Delta t}{d_{11}}\pm\left[\left(\frac{e_{11}\Delta t}{d_{11}}\right)^2-\frac{4(d_{11}+d_{12})}{C}\right]^{\frac{1}{2}}\right\} \tag{14}$$

代入（11）可求出相应的K_2解为：

$$K_2^{(1)}=-K_1^{(1)}$$
$$K_2^{(2)}=-K_1^{(3)}$$
$$K_2^{(3)}=-K_1^{(2)}$$

为了便于比较，曾取文献[1]、[2]所给出的数据，用（11)、(12）计算了$(0/0/0/90)_T$与$(0/0/90/0)_T$铺层情形，用（14）、（15）计算了$(0/0/90/90)_T$铺层情形，计算结果与文献[1]、[2]的结果非常一致。

实际上，作为工程应用最关心的是矩形板呈近似圆柱形变形时的曲率值。对此可导出更简单的计算公式。一般说，C是一个很大的数，因而由（11）、（12）略去小项，得到近似圆柱形变形时的板曲率为：

$$\begin{cases}K_1\approx\dfrac{e_{11}\Delta t}{d_{11}}\\K_2\approx0\end{cases}\quad\text{或}\begin{cases}K_1\approx0\\K_2\approx\dfrac{e_{22}\Delta t}{d_{22}}\end{cases} \tag{17}$$

相应这两组解的板势能V分别为：

$$\frac{V}{4abh}=-\frac{1}{2}d_{11}K_1^2-e_{11}K_1\Delta t=-\frac{1}{2}\frac{(e_{11}\Delta t)^2}{d_{11}}$$

或

$$\frac{V}{4abh}=-\frac{1}{2}d_{22}K_2^2-e_{22}K_2\Delta t=-\frac{1}{2}\frac{(e_{22}\Delta t)^2}{d_{22}} \tag{17}$$

利用 (16)、(17)，很容易地算出结果如表 1。由表可知，$(0/0/0/90)_T$ 与 $(0/0/90/0)_T$ 铺层情形只有一个稳定解，即势能为最小的解，它们分别为：

$$\kappa_x=0,\qquad \kappa_y=-11.34\left(\frac{1}{m}\right),$$

$$\kappa_x=0;\qquad \kappa_y=-7.00\left(\frac{1}{m}\right),$$

对于 $(0/0/90/90)_T$ 与 $(0/90/0/90)_T$ 铺层情形，则两个解相应的势能值相同，即两个解均为稳定解。以上结论均与文献[1]、[2] 一致，所得曲率计算值亦与文献 [1]、[2] 非常接近。

表 1　近似圆柱形变形的板曲率值

铺层方式	$\kappa_x=\frac{K_1}{h}\left(\frac{1}{m}\right)$	$\kappa_y=\frac{K_2}{h}\left(\frac{1}{m}\right)$	$\frac{V}{2abh(\Delta t)^2}$ (kg/cm^2c^2)
0/0/0/90	0	−11.34	−12277
	−1.34	0	−537.3
0/0/90/0	0	−7.00	−2528
	−0.224	0	−29.90
0/0/90/90	+5.80	0	−4590
	0	−5.80	−4590
0/90/0/90	+1.37	0	−542.7
	0	−1.37	−542.7

四、結　論

为了分析不对称正交层板的室温形状，不必采用文献 [1、[2] 的复杂计算方法，只需采用本文(11)、(12)或(14)、(15)来计算。如仅需确定板的近似圆柱形变形，则可用更简单的 (16) 来计算。

以上各个方程的形式非常简单明确，因此不仅大大减少了计算工作量，而且可用来分析各种材料参数和几何参数对层板变形的影响，适合于工程上设计和制造方面的实际应用。

参 考 文 献

[1] Hyer, M.W., J.Composite Materials, Vol.15, 1981, P.290-311

[2] Hyer, M.W., J.Composite Materials, Vol.16, 1982, P.318-340

[3] Hyer, M.W., Proc. ICCM-IV, Vol.1, 1982, P.373-380

[4] 胡海昌，弹性力学的变分原理及其应用，科学出版社，1981

agree well with what is in Ref.[3] where the optimality criteria is used.

THE SIMPLE CALCULATION METHOD FOR ROOM-TEMPERATURE SHAPE OF UNSYMMETRIC CROSS-PLY LAMINATES

Chen Liemin

Chinese Academy of Space Technology, P.O.Box 5106, Beijng, China

(Received Aug. 1985)

Abstract

The analysis for room-temperture shape of unsymmetric laminates after high-temperture curing is significant for design and manufacturing of composite materials. Hyer first calculated this room-temperture shape by using minimum potential energy theorem and large deflection theory of thin plate, but his calculation method is too compplex to be used in engineering.

In this paper simple calculation method for room-temperature shape of unsymmetric cross-ply laminates is prsesnted. By using the generalized variational principle and large deflection theory of thin plate, the fomulae for calculating the room-temperature shape are derived. Tts results are extremly consistent with the previous results by Hyer. Because the method in this psper is very clear and simple, it is convenient in engineering application.

EXPERIMENTAL STUDY ON RELAXATION BEHAVIOUR OF THREE-POINT BENDING BEAM OF GRP

Zhang Shuangyin

Institute of Mechanics, Academia Sinica, Beijing, China

(Received June 1985)

Abstract

Creep and relaxation phenomena were observed in the bending tests of pipe line system made of chopped strand mat glass fibre reinforced polyester(CSM-GRP). In order to characterize the relaxation behaviour of this kind of material, the experimental study on relaxation behaviour of three point-bending beam was carried out. The test scheme is that a beam specimen was loaded through a screw system to give a fixed deflection. Then the deflection was kept censtant and the load and bending strains were measured continually. Twelve specimens were