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层板弯曲与振动的近似分析

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摘　要

横向剪切变形对复合材料层合板弯曲与振动的影响甚大。在本文的近似分析中，假定板在弯曲时横向位移沿整个板厚为常量。横向剪切应变沿各层厚度方向也分别为常量，但各层不同。文中以特殊正交各向异性层合板为例，采用两种不同的方法建立了各层剪切应变間的关系，推演了层合板横向弯曲与振动的微分方程組及边界条件。算例表明，即使层合板的跨——厚比很小，用本文两种分析方案計算位移、应力及固有頻率，都仍具有較高的精度。

一、引　言

在复合材料层合板弯曲与振动问題中，横向剪切变形的影响甚为重要。文［1］、［2］中提出的考虑横向剪切变形的层合板理论，类似于以前对各向同性材料板的处理方法，虽然比较简单而且对板的刚度分析比经典层合理论有很大改进，然而它不能反映层合板弯曲后的断面翘曲现象，对应力计算改进不大。随后在文［3］—［8］中又提出了一些改进的理论，其中文［3］中提出的两种分层线性变形的力学模型值得重视。第一种模型中，假定层合板在弯曲后，板面内位移在板厚度方向的分布为分层线性变化，即在每一层内呈直线分布，而整个板的断面上则呈折线状分布。各层交界面上保持位移的连续性。这种模型后来在文［4］、［6］中曾被详细分析讨论过，并有算例表明它对变形与应力计算均有很好的精度。近年这种模型还被采用有限元方法求解，结果也较好。然而这种模型中未知量的个数随着层合板层数增加而增加，层数较多则求解困难。文［3］中的第二种模型，是在第一种模型上增添了条件，即要求在各层交界面上的两个剪应力分量也保持连续，这样就使各层的横向剪切应变相互联系起来，从而使未知量的个数大大減少并与层数无关。然而这种模型未被仔细研究。本文中以特殊正交各向异性层合板为例，按这种模型及另一种假定剪应力沿板厚呈抛物线分布的分层模型，推演了层合板弯曲与振动的基本方程与边界条件。文中算例表

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明，用这两种分层模型计算，精度都较高。

二、特殊正交各向异性层合板的平衡及运动方程

研究一块对称于板中面铺设的$2N+1$层的特殊正交各向异性层合板，取坐标系中OXY平面与板中面重合，各层编号、厚度见图1。

假设第i层在板弯曲后各点位移为（$i=-N,\cdots\cdots,O,\cdots N$）

$$\left.\begin{aligned} u_i &= u_{im}(x,y)+(z-z_i)\psi_{xi}(x,y)\\ v_i &= v_{im}(x,y)+(z-z_i)\psi_{yi}(x,y)\\ w_i &= w_{om}(x,y)\end{aligned}\right\} \quad (1)$$

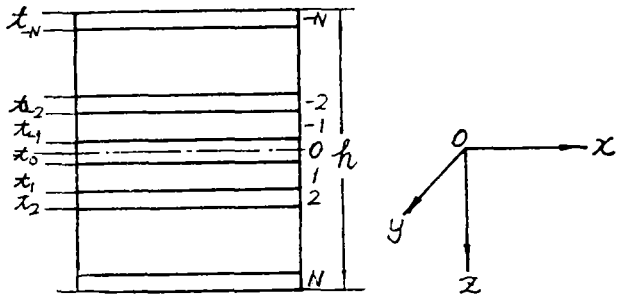

图 1

z_i、u_{im}、v_{im}分别为第i层中面的z坐标值与面内位移，ψ_{xi}、ψ_{yi}为第i层法线的转角。各层各点垂直位移均与板中面的值$w_{om}(x,y)$相同。各层内横向剪切应变为

$$\left.\begin{aligned} r_{zxi} &= \frac{\partial u_i}{\partial z}+\frac{\partial w_i}{\partial x}=\psi_{xi}+\frac{\partial w_{om}}{\partial x}\\ r_{yzi} &= \frac{\partial v_i}{\partial z}+\frac{\partial w_i}{\partial y}=\psi_{yi}+\frac{\partial w_{om}}{\partial y}\end{aligned}\right\} \quad (2)$$

各层内应力—应变关系式为（令$\sigma_z=0$）

$$\begin{Bmatrix}\sigma_{xi}\\ \sigma_{yi}\\ \tau_{xyi}\end{Bmatrix}=\begin{bmatrix}Q_{11i} & Q_{12i} & 0\\ Q_{12i} & Q_{22i} & 0\\ 0 & 0 & Q_{66i}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xi}\\ \varepsilon_{yi}\\ r_{xyi}\end{Bmatrix},\quad \begin{Bmatrix}\tau_{yzi}\\ \tau_{zxi}\end{Bmatrix}=\begin{bmatrix}Q_{44i} & 0\\ 0 & Q_{55i}\end{bmatrix}\begin{Bmatrix}r_{yzi}\\ r_{zxi}\end{Bmatrix} \quad (3)$$

其中

$$\left.\begin{aligned} Q_{11i} &= \frac{E_{xi}}{1-\nu_{xyi}\nu_{yxi}}\\ Q_{12i} &= \frac{\nu_{xyi}E_{yi}}{1-\nu_{xyi}\nu_{yxi}}=\frac{\nu_{yxi}E_{xi}}{1-\nu_{xyi}\nu_{yxi}}\\ Q_{22i} &= \frac{E_{yi}}{1-\nu_{xyi}\nu_{yxi}}\\ Q_{66i} &= G_{xyi},\ Q_{44i}=G_{yzi},\ Q_{55i}=G_{zxi}\end{aligned}\right\} \quad (4)$$

在板弯曲时，要求在第i层与第$i+1$层的交界面上，保持面内位移与剪应力连续，则有

$$\left.\begin{aligned} u_{im}+\frac{t_i}{2}\psi_{xi} &= u_{(i+1)m}-\frac{t_{i+1}}{2}\psi_{x(i+1)}\\ v_{im}+\frac{t_i}{2}\psi_{yi} &= v_{(i+1)m}=\frac{t_{i+1}}{2}\psi_{y(i+1)}\end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned} Q_{44i}\gamma_{yzi}&=Q_{44(i+1)}\gamma_{yz(i+1)}\\ Q_{55i}\gamma_{zxi}&=Q_{55(i+1)}\gamma_{zx(i+1)}\end{aligned}\right\} \tag{6}$$

(5)、(6)及(2)式使各层的面内位移间建立了联系，如用第0层中各量表示，不难求得

$$\left.\begin{aligned} u_j&=u_{0m}-(z_j-t_{xj})\frac{\partial w_{0m}}{\partial x}+t_{xj}\psi_{x0}-(z-z_j)\left[(1-\lambda_{5j})\frac{\partial w_{0m}}{\partial x}-\lambda_{5j}\psi_{x0}\right],\\ v_j&=v_{0m}-(z_j-t_{yj})\frac{\partial w_{0m}}{\partial y}+t_{yj}\psi_{y0}-(z-z_j)\left[(1-\lambda_{4j})\frac{\partial w_{0m}}{\partial y}-\lambda_{4j}\psi_{y0}\right],\\ u_{-j}&=u_{0m}+(z_j-t_{xj})\frac{\partial w_{0m}}{\partial x}-t_{xj}\psi_{x0}-(z+z_j)\left[(1-\lambda_{5j})\frac{\partial w_{0m}}{\partial x}-\lambda_{5j}\psi_{x0}\right],\\ v_{-j}&=v_{0m}+(z_j-t_{yj})\frac{\partial w_{0m}}{\partial y}-t_{yj}\psi_{y0}-(z+z_j)\left[(1-\lambda_{4j})\frac{\partial w_{0m}}{\partial y}-\lambda_{4j}\psi_{y0}\right],\end{aligned}\right\} \tag{7}$$

$$(i=1,\ 2,\ \cdots N)$$

其中

$$\lambda_{4j}=\frac{Q_{440}}{Q_{44j}},\qquad \lambda_{5j}=\frac{Q_{550}}{Q_{55j}} \tag{8}$$

其他

$$\left.\begin{aligned} t_{x1}&=\frac{1}{2}t_0+\frac{1}{2}\lambda_{51}t_1,\qquad t_{y1}=\frac{1}{2}t_0+\frac{1}{2}\lambda_{41}t_1\\ t_{xj}&=\frac{1}{2}t_0+\sum_{k=1}^{j-1}\lambda_{5k}t_k+\frac{1}{2}\lambda_{5j}t_j\\ t_{yj}&=\frac{1}{2}t_0+\sum_{k=1}^{j-1}\lambda_{4k}t_k+\frac{1}{2}\lambda_{4j}t_j\end{aligned}\right\} \tag{9}$$

这样，层合板弯曲的未知函数就仅限于第0层（中心层）中面位移u_{0m}、v_{0m}、w_{0m}及转角ψ_{x0}、ψ_{y0}，其他各层各点位移均可由(7)式确定。然而(6)式实际上将意味着板的各层剪应力相同。对于多层层合板，截面剪应力沿板厚接近抛物线状分布，所以我们也可以放弃层间剪应力连续的要求，将(6)式用下式代替：

$$\left.\begin{aligned} Q_{44i}\gamma_{yzi}&=Q_{440}\gamma_{yz0}\left[1-\left(\frac{2z_i}{h}\right)^2\right],\\ Q_{55i}\gamma_{zxi}&=Q_{550}\gamma_{zx0}\left[1-\left(\frac{2z_i}{h}\right)^2\right],\end{aligned}\right\} \tag{10}$$

这样，(7)、(9)式依然成立，只是(8)式要改成

$$\lambda_{4j}=\frac{Q_{440}}{Q_{44j}}\left[1-\left(\frac{2z_j}{h}\right)^2\right],\quad \lambda_{5j}=\frac{Q_{550}}{Q_{55j}}\left[1-\left(\frac{2z_j}{h}\right)^2\right] \tag{11}$$

本文中将λ_{4j}、λ_{5j}按(8)式或(11)式计算的两种分层模型分别称为分层（Ⅰ）或分层（Ⅱ）方案。

按照最小位能原理可以求得弯曲基本方程为

$$\left.\begin{aligned} L_{33}w_{0m}+L_{34}\psi_{x0}+L_{35}\psi_{y0}+p&=0,\\ L_{34}w_{0m}+L_{44}\psi_{x0}+L_{45}\psi_{y0}&=0,\\ L_{35}w_{0m}+L_{45}\psi_{x0}+L_{55}\psi_{y0}&=0,\end{aligned}\right\} \tag{12}$$

其中

$$
\begin{aligned}
L_{33}&=k_xG_{55}\frac{\partial^2}{\partial x^2}+k_yG_{44}\frac{\partial^2}{\partial y^2}-[(D_{11}-2D_{x11}+D_{xx11})\frac{\partial^4}{\partial x^4}+\\
&+(2D_{12}-2D_{x12}-2D_{y12}+2D_{xy12}+4D_{66}-4D_{x66}-4D_{y66}+\\
&+D_{xx66}+2D_{xy66}+D_{yy66})\frac{\partial^4}{\partial x^2\partial y^2}+\\
&+(D_{22}-2D_{y22}+D_{yy22})\frac{\partial^4}{\partial y^4}]\\
L_{34}&=k_xG_{55}\frac{\partial}{\partial x}+(D_{x11}-D_{xx11})\frac{\partial^3}{\partial x^3}+\\
&+(D_{x12}-D_{xy12}+2D_{x66}-D_{xx66}-D_{xy66})\frac{\partial^3}{\partial x\partial y^2},\\
L_{35}&=k_yG_{44}\frac{\partial}{\partial y}+(D_{y22}-D_{yy22})\frac{\partial^3}{\partial y^3}+\\
&+(D_{y12}-D_{xy12}+2D_{y66}-D_{yy66}-D_{xy66})\frac{\partial^3}{\partial x^2\partial y},\\
L_{44}&=k_xG_{55}-D_{xx11}\frac{\partial^2}{\partial x^2}-D_{xx66}\frac{\partial^2}{\partial y^2},\\
L_{45}&=-(D_{xy12}+D_{xy66})\frac{\partial^2}{\partial x\partial y},\\
L_{55}&=k_yG_{44}-D_{yy66}\frac{\partial^2}{\partial x^2}-D_{yy22}\frac{\partial^2}{\partial y^2},
\end{aligned}
\tag{13}
$$

而

$$
\begin{aligned}
&D_{pq}=Q_{pq0}J_0+2\sum_{j=1}^{N}Q_{pqj}J_j,\qquad D_{xpq}=D_{xpq0}J_0+2\sum_{j=1}^{N}Q_{pqj}J_{xj},\\
&D_{ypq}=Q_{pq0}J_0+2\sum_{j=1}^{N}Q_{pqj}J_{yj},\quad D_{xxpq}=Q_{pq0}J_0+2\sum_{j=1}^{N}Q_{pqj}J_{xxj},\\
&D_{yypq}=Q_{pq0}J_0+2\sum_{j=1}^{N}Q_{pqj}J_{yyj},\quad D_{xypq}=Q_{pq0}J_0+2\sum_{j=1}^{N}Q_{pqj}J_{xyj},\\
&\qquad\qquad(pq=11,\ 12,\ 22,\ 66)\\
&G_{44}=G_{440}t_0+2\sum_{j=1}^{N}Q_{44j}\lambda_{4j}^2t_j,\quad G_{55}=G_{550}t_0+2\sum_{j=1}^{N}Q_{55j}\lambda_{5j}^2t_j
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&J_0=\frac{1}{12}t_0^3,\quad J_j=t_jz_j^2+\frac{1}{12}t_j^3,\ J_{xj}=t_jt_{xj}z_j+\frac{1}{12}\lambda_{5j}t_j^3,\\
&J_{yj}=t_jt_{yj}z_j+\frac{1}{12}\lambda_{4j}t_j^3,\qquad J_{xxj}=t_jt_{xj}^2+\frac{1}{12}\lambda_{5j}^2t_j^3,\\
&J_{yyj}=t_jt_{yj}^2+\frac{1}{12}\lambda_{4j}^2t_j^3,\qquad J_{xyj}=t_jt_{xj}t_{yj}+\frac{1}{12}\lambda_{4j}\lambda_{5j}t_j^3
\end{aligned}
\tag{15}
$$

p 为分布载荷。层合板各种齐次边界条件为

简支$S1$（或$S2$）：$w_{0m}=0$，$M_n=0$，$M'_n=0$，$\psi_{S0}=0$（或$M_{nS}=0$）

固持$C1$（或$C2$）：$w_{0m}=0$，$\psi_{n0}=0$，$\dfrac{\partial w_{0m}}{\partial n}=0$，$\psi_{S0}=0$(或$M_{nS}=0$)

自由F：$Q_n+Q'_n+\dfrac{\partial M'_{ns}}{\partial s}=0$，$M_n=0$，$M'_n=0$，$M_{ns}=0$，

自由角点：$M'_{ns}(s-0)-M'_{ns}(s+0)=0$，

(16)

各内力用w_{0m}，ψ_{x0}、ψ_{y0}表示的表达式略。剪切修正系数k_x、k_y值在分层（Ⅰ）方案中均取$\dfrac{5}{6}$；在分层（Ⅱ）方案中则均取1，即不加修正。由(12)、(16)式可解得w_{0m}，ψ_{x0}，ψ_{y0}。通常$u_{0m}=v_{0m}=0$。故由(1)、(7)及(3)，(4)式可求出板各层面内位移与面内应力，而横向剪应力t_{zx}、t_{yz}则可按弹性力学平衡方程，由面内应力沿z向逐层求积分后求得。

采用同样变形模式及按照汉弥尔登原理可求得板的弯曲运动基本方程为

$$\left.\begin{aligned}
&L_{33}w_{0m}+L_{34}\psi_{x0}+L_{35}\psi_{y0}+p=M_{33}w_{0m}+M_{34}\psi_{x0}+M_{35}\psi_{y0},\\
&L_{34}w_{0m}+L_{44}\psi_{x0}+L_{45}\psi_{y0}\quad\ =M_{34}w_{0m}+M_{44}\psi_{x0}\quad\ ,\\
&L_{35}w_{0m}+L_{45}\psi_{x0}+L_{55}\psi_{y0}\quad\ =M_{35}w_{0m}\qquad\ +M_{55}\psi_{y0},
\end{aligned}\right\}\tag{17}$$

L算子的定义同前。M算子分别为

$$\left.\begin{aligned}
&M_{33}=\rho_s\frac{\partial^2}{\partial t^2}-(I-2I_x+I_{xx})\frac{\partial^4}{\partial t^2\partial x^2}-(I-2I_y+I_{yy})\frac{\partial^4}{\partial t^2\partial y^2},\\
&M_{34}=(I_x-I_{xx})\frac{\partial^3}{\partial t^2\partial x},\qquad M_{35}=(I_y-I_{yy})\frac{\partial^3}{\partial t^2\partial y},\\
&M_{44}=-I_{xx}\frac{\partial^2}{\partial t^2},\qquad\qquad M_{55}=-I_{yy}\frac{\partial^2}{\partial t^2}
\end{aligned}\right\}\tag{18}$$

其中

$$\left.\begin{aligned}
&\rho_s=\rho_0t_0+2\sum_{j=1}^{N}\rho_jt_j,\qquad I=\rho_0J_0+2\sum_{j=1}^{N}\rho_jJ_j,\\
&I_x=\rho_0J_0+2\sum_{j=1}^{N}\rho_jJ_{xj},\qquad I_y=\rho_0J_0+2\sum_{j=1}^{N}\rho_jJ_{yj},\\
&I_{xx}=\rho_0J_0+2\sum_{j=1}^{N}\rho_jJ_{xxj},\qquad I_{yy}=\rho_0J_0+2\sum_{j=1}^{N}\rho_jJ_{yyj}
\end{aligned}\right\}\tag{19}$$

ρ_0、ρ_j分别为板第0层及第j层的密度。各种齐次边界条件仍同(16)式，只是自由边第一个条件须改为：

$$\begin{aligned}
&Q_n+Q'_n+\frac{\partial M'_{ns}}{\partial s}+\left[(I-2I_x+I_{xx})\frac{\partial^3w_{0m}}{\partial t^2\partial x}-(I_x-I_{xx})\frac{\partial^2\psi_{x0}}{\partial t^2}\right]l+\\
&+\left[(I-2I_y+I_{yy})\frac{\partial^3w_{0m}}{\partial t^2\partial y}-(I_y-I_{yy})\frac{\partial^2\psi_{y0}}{\partial t^2}\right]m=0,
\end{aligned}\tag{20}$$

如外力$p(x,\ y,\ t)=0$，由(17)式及边界条件可求得层合板自由弯曲振动的各阶固有频率及振型。然后，就可以用振型叠加法求解强迫振动问题。对于振动问题，剪切修正系数k_x、k_y值在分层(Ⅰ)方案中均取$\dfrac{\pi^2}{12}$，在分层（Ⅱ）方案中中则仍均取1。

三、算　例

例一　四边（$x=0$，a；$y=0$，a）简支$S1$的九层［0/90/0…0/90/0］方板在载荷$p=p_0 sin(\pi x/a)sin(\pi y/a)$作用下的弯曲计算。板厚为$h$，其中0°等厚单层及90°等厚单层的总厚各为$\frac{h}{2}$。材料性能为：$E_L/E_T=25, G_{LT}/E_T=0.5, G_{TT}/E_T=0.2, \nu_{LT}=\nu_{TT}=0.25$。

引入板跨-厚比$S=a/h$及无量纲量$(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy})=(1/p_0 S^2)(\sigma_x, \sigma_y, \tau_{xy})$，$(\bar{\tau}_{zx}, \bar{\tau}_{yz})=(1/p_0 S)(\tau_{zx}, \tau_{yz})$，$\bar{w}=\pi^4 Q w/12 p_0 h S^4$。$Q=4G_{LT}+[E_L+E_T(1+2\nu_{TT})]/(1-\nu_{LT}\nu_{TL})$。为比较起见，将按本文分层（Ⅰ）、（Ⅱ）方案计算结果与经典层合理论解[12]、文[1]、[2]中的剪切变形理论解（在本例及例二中其剪切修正系数均取$\frac{5}{6}$）及文[9]三维弹性理论解结果一并列出。表1及图2表示板中点挠度$\bar{w}\left(\frac{a}{2}, \frac{a}{2}, 0\right)$值随$S$

表1　九层方板中点挠度$\bar{w}\left(\frac{a}{2}, \frac{a}{2}, 0\right)$值

S	100	50	20	10	4	2
經典层合理論	1	1	1	1	1	1
剪切变形理論	1.004	1.017	1.104	1.414	3.546	11.107
分层(Ⅰ)方案	1.005	1.021	1.130	1.513	4.083	12.922
分层(Ⅱ)方案	1.005	1.021	1.128	1.507	4.043	11.950
三維弹性理論	1.005	1.021	1.129	1.512	4.079	12.288

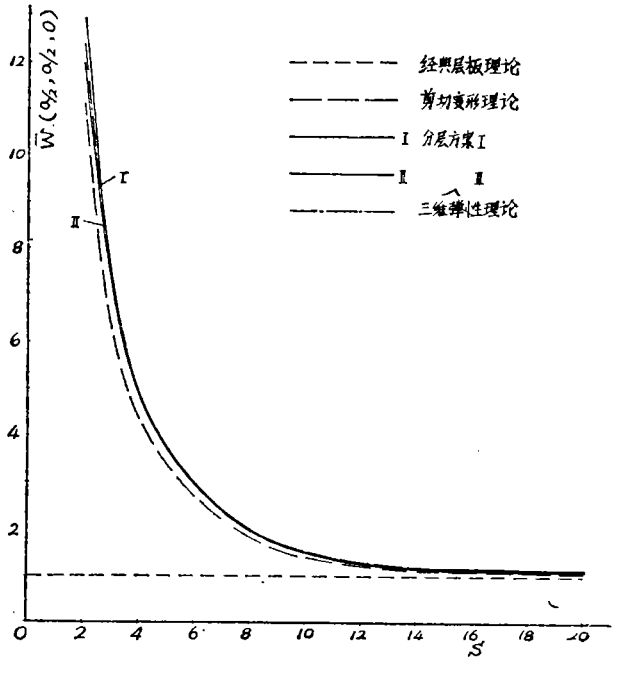

图　2

变化的情况。表 2 给出 $S=20$，10，4 的板最大应力值。此处在文[1]、[2]剪切变形理论中，横向剪应力值也由平衡方程积分求出。三维弹性理论解在 S 值较小时，板面内应力的正、负最大值的绝对值稍有差别，而且 τ_{zx}、τ_{yz} 的最大值不在 $z=0$ 处。表 2 中所标的上面的值为 $z=0$ 处值。下面的值是最大值，括号中为其对应的 z 坐标值。

表2　九层方板的最大应力值

S	理　論	$\bar{\sigma}_x$ $\left(\frac{a}{2},\frac{a}{2},\pm\frac{h}{2}\right)$	$\bar{\sigma}_y$ $\left(\frac{a}{2},\frac{a}{2},\pm\frac{2}{5}h\right)$	$\bar{\tau}_{zx}$ $\left(0,\frac{a}{2},0\right)$	$\bar{\tau}_{yz}$ $\left(\frac{a}{2},0,0\right)$	$\bar{\tau}_{xy}$ $\left(0,0,\pm\frac{h}{2}\right)$
20	剪切变形理論	±0.531	±0.440	0.256	0.223	∓0.0214
	分层(Ⅰ)方案	±0.529	±0.449	0.253	0.226	∓0.0214
	分层(Ⅱ)方案	±0.542	±0.444	0.255	0.222	∓0.0218
	三維弹性理論	±0.541	±0.444	0.255	0.221	∓0.0218
10	剪切变形理論	±0.515	±0.460	0.248	0.232	∓0.0215
	分层(Ⅰ)方案	±0.510	±0.490	0.241	0.241	∓0.0217
	分层(Ⅱ)方案	±0.556	±0.474	0.245	0.227	∓0.0232
					0.226	−0.0233
	三維弹性理論	±0.551	±0.477	0.247	0.226	+0.0235
					(−0.01h)	
4	剪切变形理論	±0.474	±0.507	0.231	0.254	∓0.0219
	分层(Ⅰ)方案	±0.509	±0.619	0.218	0.271	∓0.0220
	分层(Ⅱ)方案	±0.712	±0.604	0.210	0.222	∓0.0325
	三維弹性理論	±0.684	+0.628	0.223	0.223	−0.0337
		−0.649	−0.612	0.223	0.225	+0.0328
				(0.01h)	(−0.06h)	
任意值	經典层合理論	±0.539	±0.431	0.259	0.219	∓0.0213

例二　四边（$x=0$，a；$y=0$，b）简支 $S1$ 的三层 [0/90/0] 矩形板在载荷 $p=p_0\sin(\pi x/a)\sin(\pi y/b)$ 作用下弯曲。板厚 h，各层等厚。材性及应力无量纲量定义均与例一相同，$\bar{w}=100E_T w/p_0 hS^4$，$S=a/h$。表 3 给出 $\frac{b}{a}=1.3$ 而 $S=4$ 的计算结果。

对于 $\frac{b}{a}=1$，$S=4$ 的情况，正应力 $\bar{\sigma}_x\left(-\frac{a}{2},\frac{b}{2},\bar{z}\right)$、横向剪应力 $\bar{\tau}_{zx}\left(0,-\frac{b}{2},\bar{z}\right)$、剪应力 $\bar{\tau}_{xy}(0,0,\bar{z})$ 以及板面内位移 $\bar{u}\left(0,\frac{b}{2},\bar{z}\right)$ 沿板厚方向变化的情况分别表示在图3—6。$\bar{z}=z/h$，$\bar{u}=E_T u/p_0 hS^3$，图中不同线条类型的曲线的含意与图 2 中相同，其中三维弹性理论解同表 3 中的数据一样，都引自文[10]。为了与文[10]一致，图中 $\bar{z}$ 坐标正方

向向上。

表3　三层矩形板最大应力及中点挠度（S＝4）

$\frac{b}{a}$	理　論	$\bar{\sigma}_x$ $\left(\frac{a}{2},\frac{b}{2},\pm\frac{h}{2}\right)$	$\bar{\sigma}_y$ $\left(\frac{a}{2},\frac{b}{2},\pm\frac{h}{b}\right)$	$\bar{\tau}_{zx}$ $\left(0,\frac{b}{2},0\right)$	$\bar{\tau}_{yz}$ $\left(\frac{a}{2},0,0\right)$	$\bar{\tau}_{xy}$ $\left(0,0,\pm\frac{h}{2}\right)$	$\bar{w}$ $\left(\frac{a}{2},\frac{b}{2},0\right)$
1	經典层合理論	±0.539	±0.180	0.395	0.0823	±0.0213	0.359
	剪切变形理論	±0.437	±0.477	0.337	0.197	∓0.0369	1.480
	分层(Ⅰ)方案	±0.639	±0.721	0.246	0.274	∓0.0468	1.764
	分层(Ⅱ)方案	±0.749	±0.585	0.244	0.230	∓0.0468	1.589
	三維弹性理論	+0.801 -0.755	+0.534 -0.556	0.256 0.282 (0.27h)	0.217	-0.0511 +0.0505	——
3	經典层合理論	±0.623	±0.0252	0.440	0.0108	∓0.0083	0.503
	剪切变形理論	±0.613	±0.0934	0.436	0.0282	∓0.0205	2.363
	分层(Ⅰ)方案	±0.980	±0.129	0.352	0.0352	∓0.0282	3.122
	分层(Ⅱ)方案	±1.075	±0.105	0.333	0.0297	∓0.0261	2.667
	三維弹性理論	+1.14 -1.10	+0.109 -0.119	0.351 0.387 (0.27h)	0.0334	-0.0269 +0.0281	2.82

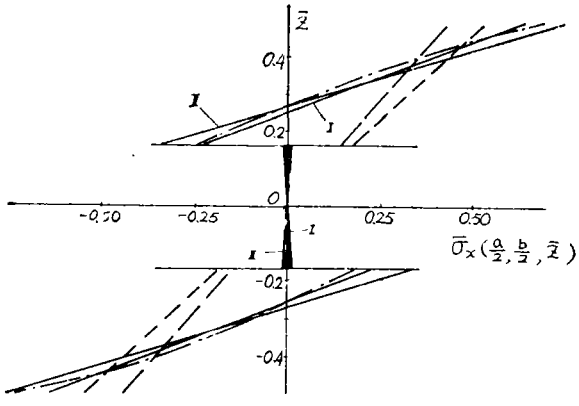

图　3

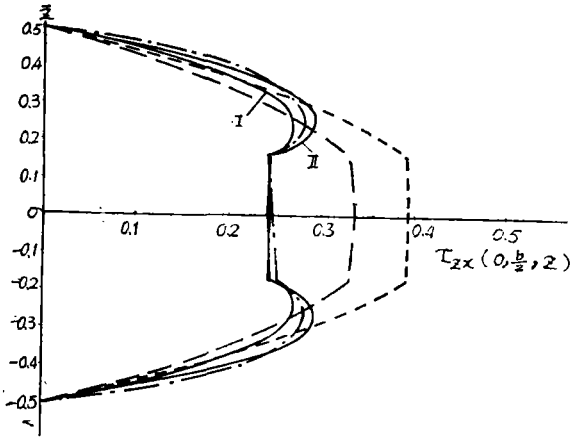

图　4

例三　边长为a的四边简支$S1$的九层[0/90/0…0/90/0]方板的固有基频计算。板各层的密度均为ρ，厚$h=a/5$，0°等厚单层及90°等厚单层的总厚各为$h/2$。材料性能为：$E_L/E_T=10$，30，40；$G_{LT}/E_T=0.6$，$G_{TT}/E_T=0.5$，$\nu_{LT}=\nu_{TT}=0.25$。引入无量纲圆频率$\lambda=10w(\rho h^2/E_T)^{1/2}$，计算结果及文[11]中三维理论差分解见表4，其中剪切变形理论的剪切修

正系数亦取$\pi^2/12$。

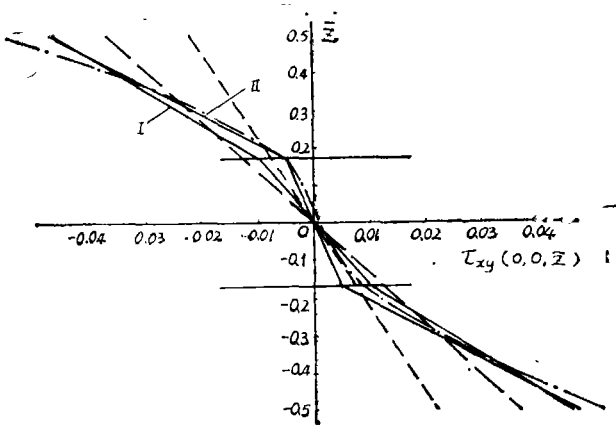

图 5

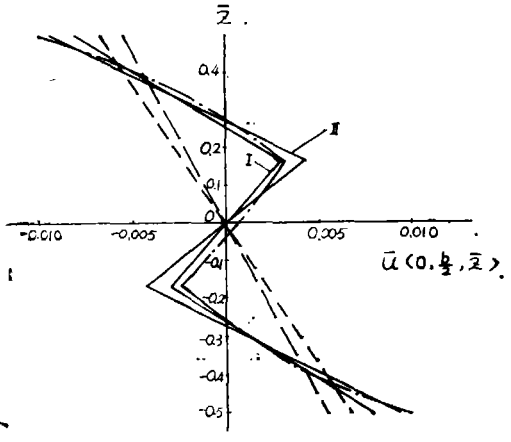

图 6

表4 九层方板固有基频λ值（S＝5）

E_L/E_T	10	30	40
經典层合理論	4.2599	6.6419	7.5565
剪切变形理論	3.4098	4.3904	4.6397
分层(Ⅰ)方案	3.4036	4.3786	4.6267
分层(Ⅱ)方案	3.4208	4.4180	4.6747
三維理論差分解	3.4432	4.4210	4.6679

四、討　論

本文以特殊正交各向异性层合板为例，讨论了考虑横向剪切变形影响的分层（Ⅰ）、（Ⅱ）两种近似分析方案，给出了它们的形式相同的基本方程与边界条件。算例表明，即使层合板的跨-厚比很小，用这两种方案计算弯曲挠度、应力及固有频率，都仍具有较高的精度。当层合板的层数较多时，剪应力沿板厚近似呈抛物线分布，故分层（Ⅰ）方案的计算结果更好。本文的工作有待推广到一般的对称、反对称乃至非对称层合板计算，以及采用有限元方法对不规则形状与复杂边界条件的层合板进行数值计算。

参 考 文 献

[1] P.C.Yang, C.H.Norris, Y.Stavsky, Int.J.Solid Structures, 2, (1966), 665—684.

[2] J.M.Whitney, N.J. Pagano, J. Appl. Mech. 37, (1970), 1031—1036.

[3] C.T.Sun, J.M.Whitney, AIAA J. 11, 2(1973) 178—183.

[4] S.Srinivas, J. Sound Vibrat. 30, 4 (1973) , 495—507.

[5] K.H.Lo, R.M. Christensen, E.M. Wu, J.Appl. Mech. 44, 4(1977), 669—676.

[6] P.Seide, An improved approximate theory for the bending of laminated plates, Mechanics Today, Vol.5, ed.by S. Nemat-Nasser, Pergamon Press (1980), 451—466.
[7] 梁继光，固体力学学报，1984年第1期，49-63.
[8] J.N. Reddy, J.Appl. Mech.51, 4 (1984), 745-752.
[9] N.J. Pagano, S.J. Hatfield, AIAA J. 10, 7 (1972) 931—933.
[10] N.J. Pagano, J.Composite Mater. 4, 1 (1970) 20—34.
[11] A.K. Noor, AIAA J. 11, 7 (1973) , 1038—1039.
[12] F. Reissner, Y Stavsky, J. Appl Mech.28, (1961) 402—408.

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ABSTRACT

With the aid of the basic theorems of Fourier transformation and Fourier series, the general solution of a pair of edge cracks at the interfaces of four different media in a rectangular sheet under longitudinal shear stress is found in this paper. It is of interest to note that the stress intensity factor of this problem is independent of the material constants of the composite rectangular sheet. The conclusion is no doubt of great importance to designers and experimenters because there is no need to carry out unnecessary tests.

STRESS INTENSITY FACTOR OF A CRACK AT THE BI-MATERIAL INTERFACE

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ABSTRACT

An analysis using complex exponential series is developed for the problem of a crack on the boundary between two dissimilar materials. The eigenvalues and the relations among the complex constants are determined by the use of crack face boundary conditions and the continuity conditions at the interface. From these results the stress and displacement fields and the stress intensity factors have been determined. the stress function used by Lin[1] and by Lin and Mar [2] are modified and the solution reduces to results obtained by Williams [6] and by Sih and Rice [3, 4]. A singular element based on the results is used to develop a finite element analysis of finite tension sheet with a central crack on a straight interface boundary.

APPROXIMATE ANALYSIS OF THE BENDING AND VIBRATION OF LAMINATES

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ABSTRACT

Transverse shear effects are important for the bending and vibration of laminated

plates. When a laminate is in bending, it is assumed in this paper that the transverse displacement is constant through the plate thickness, and the in-plane displacements vary linearly through each layer, i.e., they are piecewise linear through the plate thickness. The latter means that the transverse shear strains within each layer area assumed to be different each other. There are two methods to relate the transverse shear strains within each layer: (1) Demand the continuity of two shear stress components at all interfaces. It is equivalent to that transverse shear stresses are constant respectively through the plate thickness. (2) Assume that transverse shear stresses vary parabolically through the plate thickness. These two methods correspond with two models of the displacements of the laminate, called Piecewise (1) and Piecewise (2) schemes respectively in this paper. According to these two schemes, the governing equations for the bending of a specially orthotropic laminate under a lateral load are obtained on the basis of the principle of minimum potential energy. The unknown functions are W_{om}, ψ_{xo} and ψ_{yo} which are the transverse displacement and rotational angles of normals to the central layer of the laminate respectively. After W_{om}, ψ_{xo} and ψ_{yo} have been solved the in-plane displacements and stresses within each layer can be further obtained. Thereafter the transverse shear stresses can be given by integrating the equilibrium equations in elasticity. As for vibration problems, the similar governing equations and the boundary conditions of the laminate are derived from Hamilton's principle. Three examples are calculated. They indicate that a good estimate of displacements, stresses and natural frequencies of laminates can be given in accordance with the two schemes in this paper, even though the span-to-depth ratios of laminates are small. When a laminate is consisted of a large number of layers, the results calculated according to Piecewise (2) scheme may be bettter.

A SEMI-ANALYTICAL METHOD FOR THE BENDING OF CROSS-PLY LAMINATES

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ABSTRACT

A series expression of deflection is presented which exactly satisfies the foundation equation of bending of symmetric cross-ply laminates, and the corresponding expressions of displacements, deformations and internal forces are deiived in explicit form. The simplicity of the series makes all the derivation and the expressions simple.

In general cases, the boundary conditions of the laminate can not be satisfied exactly