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复合材料任意壳体稳定性研究(Ⅱ)——本构关系及稳定方程

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摘 要 文中分析了复合材料单层板的弹性主方向及非弹性主方向的弹性特性,进而分析了一般层合板的弹性特性及多层壳的弹性特性,以及导得具有普遍性的多层壳的稳定方程。

关键词 复合材料多层壳,本构关系,壳的稳定方程

1 引 言

对复合材料任意壳体而言,其层合板是各向异性的,又由于板厚方向的非均质性而导致应力会在层间处发生突变,因而使得壳体的本构关系及稳定方程都非常复杂。本文旨在详细分析单层板弹性方向及非弹性方向的弹性特性,并在此基础上进而分析层板的弹性性质,并借助强度分析的方法以及在《复合材料任意壳体稳定性研究(Ⅰ)——几何方程的新研究》一文的基础上推导复合材料多层壳的弹性特性,进而导致稳定性方程。而该方程对复合材料层壳具有普遍意义,具有最大的普遍性。

2 各向异性体的本构方程

2.1 正交异性弹性体的本构方程

众所周知,在均质弹性体内,若每一点沿不同方向都 具有不同的弹性特性时,其独立的弹性系数有 21 个。这是一般各向异性弹性体。但对于正交各向异性体而言,由于有三个相互正交的平面为弹性对称面,则使 弹性系数的独立个数大大减少,且其中多个为零。即,正交异性弹性体的本构方程为

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{bmatrix}$$

或写成

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$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{13} & 0 & 0 & 0\\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0\\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & S_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & S_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix}$$

可定义

$$S:=\begin{vmatrix}S_{11} & S_{12} & S_{13}\\ S_{12} & S_{22} & S_{23}\\ S_{13} & S_{23} & S_{33}\end{vmatrix}=S_{11}S_{22}S_{33}+2S_{12}S_{23}S_{13}-S_{11}S_{23}^2-S_{22}S_{13}^2-S_{33}S_{12}^2$$

而有：

$$C_{11}=\frac{S_{22}S_{33}-S_{23}^2}{S}\qquad C_{22}=\frac{S_{33}S_{11}-S_{13}^2}{S}\qquad C_{44}=\frac{1}{S_{44}}$$

$$C_{12}=\frac{S_{13}S_{23}-S_{12}S_{33}}{S}\qquad C_{23}=\frac{S_{12}S_{13}-S_{23}S_{11}}{S}\qquad C_{55}=\frac{1}{S_{55}}$$

$$C_{13}=\frac{S_{12}S_{23}-S_{13}S_{22}}{S}\qquad C_{33}=\frac{S_{11}S_{22}-S_{12}^2}{S}\qquad C_{66}=\frac{1}{S_{66}}$$

2.2 单层板弹性主方向的弹性特性

单层板的单向板纤维主向为 L 方向(即纵向)，面内垂直纤维方向为 T 方向(即横向)。对于双向板，记编织纤维的经向为 L 方向，记纬向为 T 方向。n 为单层板的法线方向。

从宏观上看，单层板是正交异性体。L、T、n 为其弹性主方向。

对于单层板，我们可以只考虑平面应力的状态，即 $\sigma_n=\tau_{Tn}=\tau_{nL}=0$。即

$$\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \varepsilon_n\\ \gamma_{Tn}\\ \gamma_{nL}\\ \gamma_{LT}\end{bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{13} & 0 & 0 & 0\\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0\\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & S_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & S_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_L\\ \sigma_T\\ 0\\ 0\\ 0\\ \tau_{LT}\end{bmatrix}$$

将面内部分分离出来，成为

$$\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \gamma_{LT}\end{bmatrix}=\begin{bmatrix}S_{11} & S_{12} & 0\\ S_{12} & S_{22} & 0\\ 0 & 0 & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{bmatrix},\quad \text{将其求逆，可得}\begin{bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \gamma_{LT}\end{bmatrix}$$

式中：

$$Q_{11}=\frac{S_{22}}{S_{11}S_{22}-S_{12}^2},\ Q_{12}=\frac{S_{12}}{S_{11}S_{22}-S_{12}^2},\quad Q_{22}=\frac{S_{11}}{S_{11}S_{22}-S_{12}^2},\ Q_{66}=\frac{1}{S_{66}}$$

$$S_{11}=\frac{Q_{22}}{Q_{11}Q_{22}-Q_{12}^2},\ S_{12}=-\frac{Q_{12}}{Q_{11}Q_{22}-Q_{12}^2},\ S_{22}=\frac{Q_{11}}{Q_{11}Q_{22}-Q_{12}^2},\ S_{66}=\frac{1}{Q_{66}}$$

因为 $S_{11}S_{22}-S_{12}^2=SC_{33}$；同时，若令 $CS=1$，则有 $S_{22}=\dfrac{C_{33}C_{11}-C_{13}^2}{C}$

因而可以导得：

$$Q_{11}=\frac{S_{22}}{S_{11}S_{22}-S_{12}^2}=\frac{C_{33}C_{11}-C_{13}^2}{CSC_{33}}=C_{11}-\frac{C_{13}C_{13}}{C_{33}}$$

$$Q_{22}=\frac{S_{11}}{S_{11}S_{22}-S_{12}^2}=\frac{C_{33}C_{22}-C_{23}^2}{CSC_{33}}=C_{22}-\frac{C_{23}C_{23}}{C_{33}}$$

$$Q_{12}=-\frac{S_{12}}{S_{11}S_{22}-S_{12}^2}=\frac{C_{12}C_{33}-C_{13}C_{23}}{CSC_{33}}=C_{12}-\frac{C_{13}C_{23}}{C_{33}}$$

因为对于正交各向异性材料有　　$C_{16}=C_{26}=C_{36}=0$

故有　　$Q_{16}=Q_{26}=0$；　　$Q_{66}=C_{66}$

因而有

$$Q_{ij}=C_{ij}-\frac{C_{i3}C_{j3}}{C_{33}}\qquad(i,j=1,2,6)\tag{A}$$

即，二维刚度系数 Q_{ij} 与三维刚度系数 C_{ij} 存在关系式(A)。称 Q_{ij} 为折减刚度系数(reduced stiffnesses)。折减刚度系数与工程常数的关系式为：

$$Q_{11}=\frac{E_L}{1-\nu_{LT}\nu_{LT}}\qquad Q_{12}=\frac{E_L\nu_{TL}}{1-\nu_{LT}\nu_{TL}}=\frac{E_T\nu_{LT}}{1-\nu_{LT}\nu_{TL}}$$

$$Q_{22}=\frac{E_T}{1-\nu_{LT}\nu_{TL}}\qquad Q_{66}=G_{LT}$$

式中 E_L 为纵向弹性模量，E_T 为横向弹性模量，G_{LT} 为横向剪切模量，ν_{LT} 是纵向(L 向)伸缩时决定横向(T 向)缩伸 的泊松比，ν_{TL} 是横向(T 向)伸缩时决定纵向(L 向)缩伸 的泊松比。

由马克斯韦尔定理可知：　　$E_L\nu_{TL}=E_T\nu_{LT}$

在这里，垂直于弹性对称面的方向为弹性主方向，坐标轴在弹性主方向时称为弹性主轴。平面应力状态下的单层板有四个独立的弹性常数 E_L、E_T、$\nu_{TL}(\nu_{LT})$、G_{LT}。

3.3　单层板非弹性主方向的弹性特性

设 参考轴系 x、y 与单层板弹性主方向坐标系 L、T 在面内相差一个 α 角，并规定从 x 正方向逆时针转到 L 方向时的 α 角为正。

非弹性主方向的刚度矩阵 $[\overline{Q}]$ 与弹性主方向的刚度矩阵 $[Q]$ 的关系为：

$$[\overline{Q}]=[T]_\sigma[Q][T]_\sigma^T$$

展开后，得 $\overline{Q}_{ij}$　　(i,j=1,2,6)

$$\overline{Q}_{11}=Q_{11}\cos^4\alpha+2(Q_{12}+2Q_{66})\sin^2\alpha\cos^2\alpha+Q_{22}\sin^4\alpha$$

$$\overline{Q}_{12}=Q_{11}+(Q_{11}+Q_{22}-2Q_{12}-4Q_{66})\sin^2\alpha\cos^2\alpha$$

$$\overline{Q}_{22}=Q_{12}\cos^4\alpha+2(Q_{12}+2Q_{66})\sin^2\alpha\cos^2\alpha+Q_{11}\sin^4\alpha$$

$$\overline{Q}_{22}=Q_{12}\cos^4\alpha+2(Q_{12}+2Q_{66})\sin^2\alpha\cos^2\alpha+Q_{11}\sin^4\alpha$$

$$\overline{Q}_{66}=Q_{66}+(Q_{11}+Q_{22}-2Q_{12}-4Q_{66})\sin^2\alpha\cos^2\alpha$$

$$\overline{Q}_{16}=(Q_{11}-Q_{12}-2Q_{66})\sin\alpha\cos^3\alpha-(Q_{22}-Q_{12}-2Q_{66})\sin^3\alpha\cos\alpha$$

$$\overline{Q}_{26}=(Q_{11}-Q_{12}-2Q_{66})\sin^3\alpha\cos\alpha-(Q_{22}-Q_{12}-2Q_{66})\sin\alpha\cos^3\alpha$$

在平面应力状态下，单层板非弹性主方向的柔度矩阵和是刚度矩阵虽然都是满阵，但是独立的弹性系数仍然是四个。

在计算单层板非弹性主方向的刚度系数 $\overline{Q}_{ij}$ 时，引入下述不变量刚度是方便的：

$$U_1=\frac{1}{8}(3Q_{11}+3Q_2+2Q_{12}+4Q_{66});\quad U_2=\frac{1}{2}(Q_{11}-Q_{22})$$

$$U_3=\frac{1}{8}(Q_{11}+Q_{22}-2Q_{12}-4Q_{66});\quad U_4=\frac{1}{8}(Q_{11}+Q_{22}+6Q_{12}+4Q_{66})$$

$$U_5=\frac{1}{8}(Q_{11}+Q_{22}-2Q_{12}+4Q_{66})$$

因此有：

$$\overline{Q}_{11}=U_1+U_2\cos2\alpha+U_3\cos4\alpha;\quad \overline{Q}_{12}=U_4-U_3\cos4\alpha$$

$$\overline{Q}_{22}=U_1-U_2\cos2\alpha+U_3\cos4\alpha;\quad \overline{Q}_{66}=U_5-U_3\cos4\alpha$$

$$\overline{Q}_{16}=\frac{1}{2}U_2\sin2\alpha+U_3\sin4\alpha;\quad \overline{Q}_{26}=\frac{1}{2}U_2\sin2\alpha-U_3\sin4\alpha$$

上试的优点在于不含三角函数的乘方而便于计算，也容易对 Q_{ij} 随 α 角的变化作出判断。

非弹性主方向的弹性系数($\overline{S}_{ij}$,$\overline{Q}_{ij}$)、工程常数只有四个是独立的，五个不变量刚度也只有四个是独立的。

3 一般层合板的弹性特性

层合板的内力素可表达为：

$$N_x=\int_{-h}^{h}\sigma_x dz,\quad N_y=\int_{-h}^{h}\sigma_y dz,\quad N_{xy}=\int_{-h}^{h}\tau_{xy}dz=N_{yx}$$

$$Q_x=\int_{-h}^{h}\tau_{xz}dz,\quad Q_y=\int_{-h}^{h}\tau_{yz}dz$$

$$M_x=\int_{-h}^{h}\sigma_x z dz,\quad M_y=\int_{-h}^{h}\sigma_y z dz,\quad M_{xy}=\int_{-h}^{h}\tau_{xy}z dz=M_{yx}$$

而内力素与变形的关系为：

$$\begin{bmatrix}N_x\\N_y\\N_{xy}\\M_x\\M_y\\M_{xy}\end{bmatrix}\begin{bmatrix}A_{11}&A_{12}&A_{16}&B_{11}&B_{12}&B_{16}\\A_{12}&A_{22}&A_{26}&B_{12}&B_{22}&B_{26}\\A_{16}&A_{26}&A_{66}&B_{16}&B_{26}&B_{66}\\B_{11}&B_{12}&B_{16}&D_{11}&D_{12}&D_{16}\\B_{12}&B_{22}&B_{26}&D_{12}&D_{22}&D_{26}\\B_{16}&B_{26}&B_{66}&D_{16}&D_{26}&D_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\gamma_{xy}\\\chi_x\\\chi_y\\\chi_{xy}\end{bmatrix}$$

或

$$\begin{bmatrix}N\\M\end{bmatrix}=\begin{bmatrix}A&B\\B&D\end{bmatrix}\begin{bmatrix}\varepsilon\\\chi\end{bmatrix}$$

式中

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

$$\chi_x=\frac{\partial^2 w}{\partial x^2},\quad \chi_y=-\frac{\partial^2 w}{\partial y^2},\quad \chi_{xy}=-2\frac{\partial^2 w}{\partial x\partial y}$$

设层合板由 N 层任意铺设的单层板所构成。第 k 层本构方程为

$$\begin{bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{bmatrix}^k=\begin{bmatrix}\overline{Q}_{11}&\overline{Q}_{12}&\overline{Q}_{16}\\\overline{Q}_{12}&\overline{Q}_{22}&\overline{Q}_{26}\\\overline{Q}_{16}&\overline{Q}_{26}&\overline{Q}_{66}\end{bmatrix}^k\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\gamma_{xy}\end{bmatrix}^k$$

故有　$\{N\} = \sum_{k=1}^{N}\int_{Z_{k-1}}^{Z_k}[\overline{Q}]^k\{\varepsilon\}dZ;\qquad \{M\} = \sum_{k=1}^{N}\int_{Z_{k-1}}^{Z_k}[\overline{Q}]^k\{\varepsilon\}ZdZ$

及

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{N}\overline{Q}_{ij}^{(k)}(Z_k - Z_{k-1}) \\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{N}\overline{Q}_{ij}^{(k)}(Z_k^2 - Z_{k-1}^2) \\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{N}\overline{Q}_{ij}^{(k)}(Z_k^3 - Z_{k-1}^3) \end{aligned}\right\}\qquad i,j = 1,2,6$$

层合板可以是由不同材质的单层板所构成，也可以是由不同铺设方向的相同材质的各向异性单层板所构成。无论是前者还是后者，在层合板的厚度方向上都具有宏观非均质性。这种非均质性使层合板的力学分析变得复杂。例如，在一般情形中，面内内力可以引起弯（扭）曲变形，而弯（扭）曲内力可以引起面内变形，发生所谓耦合效应。层合板通常是各向异性的，而且由于铺层情形的多样性，使层合板未必具有确定的主方向。采用直法线不变假定，使层合板的变形分析化为中面的变形分析。应当注意，由于层合板厚度方向的非均质性，应力会在层间处发生突变。内力素的积分式可以化为分层积分再求和的形式。

4　复合材料多层壳的弹性特性

在正交曲线坐标系中，对于任意形状壳体有：

$$N_{11} = \int_{-h}^{h}\sigma_{11}(1 + \frac{Z}{R_2})dZ \qquad M_{11} = \int_{-h}^{h}\sigma_{11}(1 + \frac{Z}{R_2})ZdZ$$

$$N_{22} = \int_{-h}^{h}\sigma_{22}(1 + \frac{Z}{R_1})dZ \qquad M_{22} = \int_{-h}^{h}\sigma_{22}(1 + \frac{Z}{R_1})ZdZ$$

$$N_{12} = \int_{-h}^{h}\sigma_{12}(1 + \frac{Z}{R_2})dZ \qquad M_{12} = \int_{-h}^{h}\sigma_{12}(1 + \frac{Z}{R_2})ZdZ$$

$$N_{21} = \int_{-h}^{h}\sigma_{21}(1 + \frac{Z}{R_1})dZ \qquad M_{21} = \int_{-h}^{h}\sigma_{21}(1 + \frac{Z}{R_1})ZdZ$$

其中变形后的应变张量为

$$\hat{f}_{11} = e_{11} + Z\rho_{11};\qquad \hat{f}_{22} = e_{22} + Z\rho_{22};\qquad \hat{f}_{12} = e_{12} + Z\rho_{12}$$

式中

$$e_{11} = l_{11} + \frac{1}{2}(l_{11}^2 + l_{21}^2 + l_{31}^2);\quad e_{22} = l_{22} + \frac{1}{2}(l_{22}^2 + l_{12}^2 + l_{32}^2)$$

$$e_{12} = \frac{1}{2}(l_{12} + l_{21}) + l_{11}l_{12} + l_{21}l_{22} + l_{31}l_{32};\ \rho_{11} = \hat{m}_{11} - \frac{l_{11}}{R_1};\quad \rho_{22} = \hat{m}_{22} - \frac{l_{22}}{R_2}$$

$$\begin{aligned} \rho_{12} &= \hat{m}_{12} - \frac{l_{12}}{2}(\frac{1}{R_1} + \frac{1}{R_2}) - \frac{l_{21}}{2}(\frac{1}{R_1} - \frac{1}{R_2}) \\ &= \hat{m}_{21} - \frac{l_{21}}{2}(\frac{1}{R_2} + \frac{1}{R_1}) - \frac{l_{12}}{2}(\frac{1}{R_2} - \frac{1}{R_1}) = \hat{m}_{12} - \frac{1}{R_1}\frac{l_{12} + l_{21}}{2} - \frac{1}{R_2}\frac{l_{12} - l_{21}}{2} \\ &= \hat{m}_{21} - \frac{1}{R_2}\frac{l_{21} + l_{12}}{2} - \frac{1}{R_1}\frac{l_{21} - l_{12}}{2} = \frac{1}{2}(\hat{m}_{12} - \frac{l_{21}}{R_1} + \hat{m}_{21} - \frac{l_{12}}{R_2}) = \rho_{21} \end{aligned}$$

$$l_{11} = \frac{1}{A_1}u_{1,1} + u_2\frac{A_{1,2}}{A_1A_2} + \frac{w}{R_1};\quad l_{12} = \frac{1}{A_2}u_{1,2} - u_2\frac{A_{2,1}}{A_2A_1};\quad l_{21} = \frac{1}{A_1}u_{2,1} - u_1\frac{A_{1,2}}{A_1A_2}$$

$$l_{22} = \frac{1}{A_2}u_{2,2} + u_1\frac{A_{2,1}}{A_2A_1} + \frac{w}{R_2};\quad l_{31} = \frac{1}{A_1}w_{,1} - \frac{u_1}{R_1};\qquad l_{32} = \frac{1}{A_2}w_{,2} - \frac{u_2}{R_2}$$

$$-\hat{m}_{11}=\frac{1}{A_1}l_{31,1}+l_{32}\frac{A_{1,2}}{A_1A_2};\qquad -\hat{m}_{22}=\frac{1}{A_2}l_{32,2}+l_{31}\frac{A_{2,1}}{A_2A_1}$$

$$-\hat{m}_{12}=\frac{1}{A_2}l_{31,2}+l_{32}\frac{A_{2,1}}{A_2A_1};\qquad -\hat{m}_{21}=\frac{1}{A_1}l_{32,1}+l_{31}\frac{A_{1,2}}{A_1A_2}$$

$$l_{k\beta}=u_{(k|\beta)};\quad -\hat{m}^{C}_{\alpha\beta}=(u_{(3|\alpha)}||_{\beta});\quad \rho_{\alpha\beta}=-u_3|_{\alpha\beta}+\varepsilon_{\alpha\gamma}b^{\gamma}_{\beta}+\varepsilon_{\beta\gamma}b^{\gamma}_{\alpha}$$

$$A_{ij}=\int_{-h}^{h}\overline{Q}_{ij}\mathrm{d}Z;\quad B_{ij}=\int_{-h}^{h}\overline{Q}_{ij}Z\mathrm{d}Z;\quad D_{ij}=\int_{-h}^{h}\overline{Q}_{ij}Z^2\mathrm{d}Z;\quad E_{ij}=\int_{-h}^{h}\overline{Q}_{ij}Z^3\mathrm{d}Z$$

综合上述公式，可知：

$$\begin{aligned}N_{11}&=\int_{-h}^{h}\sigma_{11}(1+\frac{z}{R_2})\mathrm{d}z\\&=\int_{-h}^{h}[\overline{Q}(e_{11}+z\rho_{11})+\overline{Q}_{12}(e_{22}+z\rho_{22})+\overline{Q}_{16}(e_{12}+z\rho_{12})](1+\frac{z}{R_2})\mathrm{d}z\\&=(A_{11}+\frac{B_{11}}{R_2})e_{11}+(A_{12}+\frac{B_{12}}{R_2})e_{22}+(A_{16}+\frac{B_{16}}{R_2})e_{12}\\&\quad+(B_{11}+\frac{D_{11}}{R_2})\rho_{11}+(B_{12}+\frac{D_{12}}{R_2})\rho_{22}+(B_{16}+\frac{D_{16}}{R_2})\rho_{12}\end{aligned}$$

$$\begin{aligned}N_{22}&=\int_{-h}^{h}\sigma_{22}(1+\frac{z}{R_2})\mathrm{d}z\\&=(A_{22}+\frac{B_{22}}{R_1})e_{22}+(A_{21}+\frac{B_{21}}{R_1})e_{11}+(A_{26}+\frac{B_{26}}{R_1})e_{21}\\&\quad+(B_{22}+\frac{D_{22}}{R_1})\rho_{22}+(B_{21}+\frac{D_{21}}{R_1})\rho_{11}+(B_{26}+\frac{D_{26}}{R_1})\rho_{21}\end{aligned}$$

$$\begin{aligned}N_{12}&=\int_{-h}^{h}\sigma_{12}(1+\frac{z}{R_2})\mathrm{d}z\\&=(A_{61}+\frac{D_{61}}{R_2})\rho_{11}+(B_{62}+\frac{D_{62}}{R_2})\rho_{22}+(B_{66}+\frac{D_{66}}{R_2})\rho_{12}+(B_{61}+\frac{D_{61}}{R_2})\rho_{11}\\&\quad+(B_{62}+\frac{D_{62}}{R_2})\rho_{22}+(B_{66}+\frac{D_{66}}{R_2})\rho_{12}\end{aligned}$$

将 N_{12}中的 R_2 换成 R_1 即得 N_{21}的表达式。同法也可得到 M_{11}、M_{22}、$M_{12}(M_{21})$的表达式。即

$$\begin{bmatrix}N_{11}\\N_{22}\\N_{12}\\M_{11}\\M_{22}\\M_{12}\end{bmatrix}=\begin{bmatrix}A_{11}+\frac{B_{11}}{R_2} & A_{12}+\frac{B_{12}}{R_2} & A_{16}+\frac{B_{16}}{R_2} & B_{11}+\frac{D_{11}}{R_2} & B_{12}+\frac{D_{12}}{R_2} & B_{16}+\frac{D_{16}}{R_2}\\A_{21}+\frac{B_{21}}{R_1} & A_{22}+\frac{B_{22}}{R_1} & A_{26}+\frac{B_{26}}{R_1} & B_{21}+\frac{D_{21}}{R_1} & B_{22}+\frac{D_{22}}{R_1} & B_{26}+\frac{D_{26}}{R_1}\\A_{61}+\frac{B_{61}}{R_2} & A_{62}+\frac{B_{62}}{R_2} & A_{66}+\frac{B_{66}}{R_2} & B_{61}+\frac{D_{61}}{R_2} & B_{62}+\frac{D_{62}}{R_2} & B_{66}+\frac{D_{66}}{R_2}\\B_{11}+\frac{D_{11}}{R_2} & B_{12}+\frac{D_{12}}{R_2} & B_{16}+\frac{D_{16}}{R_2} & D_{11}+\frac{E_{11}}{R_2} & D_{12}+\frac{E_{12}}{R_2} & D_{16}+\frac{E_{16}}{R_2}\\B_{21}+\frac{D_{21}}{R_1} & B_{22}+\frac{D_{22}}{R_1} & B_{26}+\frac{D_{26}}{R_1} & D_{21}+\frac{E_{21}}{R_1} & D_{22}+\frac{E_{22}}{R_1} & D_{26}+\frac{E_{26}}{R_1}\\B_{61}+\frac{D_{61}}{R_2} & B_{62}+\frac{D_{62}}{R_2} & B_{66}+\frac{D_{66}}{R_2} & D_{61}+\frac{E_{61}}{R_2} & D_{62}+\frac{E_{62}}{R_2} & D_{66}+\frac{E_{66}}{R_2}\end{bmatrix}\begin{bmatrix}e_{11}\\e_{22}\\e_{12}\\\rho_{11}\\\rho_{22}\\\rho_{12}\end{bmatrix}\tag{B}$$

将上式之二、五行各无素的第二项乘以$\frac{1}{R_1}$，其余各行的第二项乘以$\frac{1}{R_2}$，则可写成：

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \widetilde{A}_{16} & \widetilde{B}_{11} & \widetilde{B}_{12} & \widetilde{B}_{16} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \widetilde{A}_{26} & \widetilde{B}_{21} & \widetilde{B}_{22} & \widetilde{B}_{26} \\ \widetilde{A}_{61} & \widetilde{A}_{62} & \widetilde{A}_{66} & \widetilde{B}_{61} & \widetilde{B}_{62} & \widetilde{B}_{66} \\ \widetilde{B}_{11} & \widetilde{B}_{12} & \widetilde{B}_{16} & \widetilde{D}_{11} & \widetilde{D}_{12} & \widetilde{D}_{16} \\ \widetilde{B}_{21} & \widetilde{B}_{22} & \widetilde{B}_{26} & \widetilde{D}_{21} & \widetilde{D}_{22} & \widetilde{D}_{26} \\ \widetilde{B}_{61} & \widetilde{B}_{62} & \widetilde{B}_{66} & \widetilde{D}_{61} & \widetilde{D}_{62} & \widetilde{D}_{66} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_6 \\ \rho_1 \\ \rho_2 \\ \rho_6 \end{bmatrix}$$

即

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B} & \widetilde{D} \end{bmatrix} \begin{bmatrix} e \\ \rho \end{bmatrix}$$

式中　$\widetilde{A}_{ij} \neq \widetilde{A}_{ji}$，$\widetilde{B}_{ij} \neq \widetilde{B}_{ji}$，$\widetilde{D}_{ij} \neq \widetilde{D}_{ji}$，但注意：　$A_{ij} = A_{ji}$，$B_{ij} = B_{ji}$，$D_{ij} = D_{ji}$

对于球壳，因为 $R_1 = R_2 = R$，　所以有　$\widetilde{A}_{ij} = \widetilde{A}_{ji}$　$\widetilde{B}_{ij} = \widetilde{B}_{ji}$　$\widetilde{D}_{ij} = \widetilde{D}_{ji}$

对于扁壳，因为$\frac{1}{R_1} \cong 0$，$\frac{1}{R_2} \cong 0$，　所以有　$\widetilde{A}_{ij} = A_{ij}$　$\widetilde{B}_{ij} = B_{ij}$　$\widetilde{D}_{ij} = D_{ij}$

5　复合材料多层壳的稳定性方程

设外载为静水压，则线性稳定方程为

$$N^{\alpha\delta}||_{\alpha} - Q^{\alpha} b_{\alpha}^{\delta} + \overline{N}^{\alpha k} u^{\delta}|_{k\alpha} = 0; \qquad N^{\alpha\beta} b_{\alpha\beta} + Q^{\alpha}||_{\alpha} + \overline{N}^{\alpha k} u^{3}|_{k\alpha} = 0$$

或

$$(A_2 N_{11})_{,1} + A_{1,2} N_{12} - A_{2,1} N_{22} + \frac{A_1 A_2}{R_1} Q_1 + \overline{N}_{11}[A_2 l_{11,1} + A_{1,2}(l_{12} + l_{21})]$$
$$+ \overline{N}_{22}[A_1 l_{12,2} - A_{2,1}(l_{22} - l_{11}) - \frac{A_1 A_2}{R_2} l_{31}] + (\overline{N}_{12} + \overline{N}_{21})[A_1 l_{11,2} - A_{2,1}(l_{21} + l_{12})] = 0$$

$$(A_1 N_{22})_{,2} + (A_2 N_{12})_{,1} + A_{2,1} N_{21} - A_{1,2} N_{11} + \frac{A_2 A_1}{R_2} Q_2 + \overline{N}_{22}[A_1 l_{22,2} + A_{2,1}(l_{21} + l_{12})]$$
$$+ \overline{N}_{11}[A_2 A_{21,1} - A_{1,2}(l_{11} - l_{22}) - \frac{A_2 A_1}{R_1} l_{32}] + (\overline{N}_{21} - \overline{N}_{12})[A_2 l_{22,1} - A_{1,2}(l_{12} + l_{21})] = 0$$

$$(A_2 Q_1)_{,1} + (A_1 Q_2)_{,2} - A_1 A_2 (\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}) + \overline{N}_{11}(A_2 l_{31,1} + A_{1,2} l_{32} - \frac{A_1 A_2}{R_1} l_{11})$$
$$+ \overline{N}_{22}(A_1 l_{32,2} + A_{2,1} l_{31} - \frac{A_2 A_1}{R_2} l_{22}) + (\overline{N}_{12} + \overline{N}_{21})(A_2 l_{32,1} - A_{1,2} l_{31} - \frac{A_1 A_2}{R_1} l_{12}) = 0 \qquad \text{(C)}$$

式中

$$A_1 A_2 Q_1 = (A_2 M_{11})_{,1} + (A_1 N_{21})_{,2} + A_{1,2} M_{12} - A_{2,1} M_{22}$$
$$A_2 A_1 Q_2 = (A_1 M_{22})_{,2} + (A_2 M_{12})_{,1} + A_{2,1} M_{21} - A_{1,2} M_{11}$$

微扰态应变为无穷小应变，也就是

$$e_{11} \cong \varepsilon_{11} = l_{11}, \quad e_{22} \cong \varepsilon_{22} = l_{22}, \quad e_{12} \cong \varepsilon_{12} = \frac{1}{2}(l_{12} + l_{21})$$

$$\rho_{11} = -\frac{1}{A_1} l_{31,1} - \frac{A_{1,2}}{A_1 A_2} l_{32} - \frac{l_{11}}{R_1}, \qquad \rho_{22} = -\frac{1}{A_2} l_{32,2} - \frac{A_{2,1}}{A_2 A_1} l_{31} - \frac{l_{22}}{R_2}$$

$$\rho_{12} = -\frac{1}{A_2} l_{31,2} + \frac{A_{2,1}}{A_2 A_1} l_{32} - \frac{l_{12}}{2}(\frac{1}{R_1} + \frac{1}{R_2}) - \frac{l_{21}}{2}(\frac{1}{R_1} - \frac{1}{R_2})$$

因而，将上述各式代入(B)式中，可得 $N_{11}, N_{22}, N_{12}, M_{11}, M_{22}, M_{12}, N_{21}, M_{21}$ 用 $l_{k\alpha}$ 和 $l_{k\alpha,\beta}$ 的表达式。再代入(C)式中，即可得用扰动位移梯度 $l_{k\alpha}$ 和位移二次梯度 $l_{k\alpha,\beta}$ 表达的正交曲线坐标系中任意形状壳体线性稳定方程组。

事实上，复合材料多层壳的稳定方程是非常复杂的。

6 结束语

本文提出 $\hat{f}_{\alpha\beta}=e_{\alpha\beta}+z\rho_{\alpha\beta}$， $e_{\alpha\beta}=\frac{1}{2}(u_\alpha|_\beta+u_\beta|_\alpha+u^k|_\alpha u_k|_\beta)$

以及 $\rho_{\alpha\beta}=-u_3|_{\alpha\beta}+\frac{1}{2}(u_\alpha|_\gamma+u_\gamma|_\alpha)b^\gamma_\beta+\frac{1}{2}(u_\beta|_\gamma+u_\gamma|_\beta)b^\gamma_\alpha$ 使 几何方程既简单又比较精确。导出的(B)式使 层壳的公式有别于层板的公式。而目前文献上复合材料层壳绝大部分只讨论扁壳，本文的任意形状复合材料层壳公式则具有最大的普遍性。

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ON THE STABILITY OF ARBITRARY COMPOSITES SHELLS(Ⅱ) ——CONSTITUTIVE RELATIONS AND STABILITY EQUATIONS

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Abstract In the present paper the on－axis and off－axis elasticity of composite lamina is analysed. Through analysing the elastic characteristics of composite laminated plates and shells，a new set of general stability equations of composite laminate shells is established.

Key words composite laminated shell，constitutive relations，stability