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纤维增强复合材料层合板屈曲性态分析的边界元法

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摘 要 本文用边界元法分析了纤维增强复合材料正交各向异性层合板的屈曲性态。为了克服在用边界元法求解正交各向异性层合板屈曲时寻求相应的基本解的困维，本文采用了双重傅立叶级数和引用等效荷载的概念，建立了层合板屈曲临界荷载的特征方程。算例说明了本文方法的可行性和有效性。

关键词 复合材料层合板，边界元法，基本解，屈曲性态

中图分类号 O316，TB330.01

纤维增强复合材料层合板具有强度高、重量轻、绝缘、隔热、可设计性好等许多其它结构元件无法企及的优异性能，已经在近代工业的许多重要部门得到广泛应用。由于层合板的各向异性，铺层的非对称性，使得层合板变形存在较强的拉弯、弯扭耦合效应。边界元法作为一种强有力的数值计算方法，在复杂工程结构的力学性态分析中获得成功，而且比较方便。各向异性材料问题求解的困难在于寻找基本解，而基本解在边界元法中起着非常关键的作用。到目前为止，能求得基本解的偏微分方程还只有很少几类[1]。本人在文献[2]中用边界元法分析了复合材料特殊正交各向异性层合板的振动，采用了傅立叶级数形式的近似基本解。本文用边界元法研究了工程中常用的四边简支纤维增强复合材料层合矩形板在任意横向荷载作用下的屈曲问题。为了克服寻求基本解的困维，本文采用了双重 Fourier 级数，建立层合板屈曲临界荷载的特征方程。编写了 BASIC 程序，在 IBM-PC 微机上计算了实例。计算结果表明本文方法的可行性和有效性。

1 层合板屈曲的本构方程

如图 1 所示，在笛卡尔坐标系 $o\text{-}xyz$ 中，考虑一块在 x 轴和 y 轴方向长度分别为 a 和 b，z 轴方向厚度为 h 的由若干层铺层迭合而成的纤维增强层合板，不计横向剪切变形影响、层合板在横向荷载作用下屈曲时，每层铺层都处于平面应力

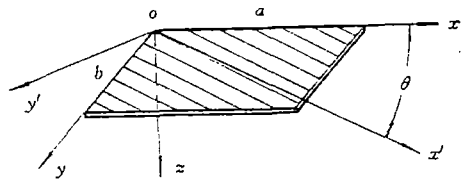

图 1 层合板的铺层和坐标系

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状态。设 x'、y' 为铺层材料的主方向，沿材料主方向铺层是正交各向异性的，于是此板第 k 层材料的应力应变关系为

$$\{\sigma\} = [H]\{\varepsilon\} \tag{1}$$

式中：

$$[H] = \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \tag{2}$$

为正交各向异性铺层在总体坐标 xoy 下的弹性系数矩阵，矩阵中每项元素的具体形式为

$$\left.\begin{aligned}
H_{11} &= C_{11}\cos^4\theta + 2(C_{12} + 2C_{66})\sin^2\theta\cos^2\theta + C_{22}\sin^4\theta \\
H_{12} &= (C_{11} + C_{22} - 4C_{66})\sin^2\theta\cos^2\theta + C_{12}(\sin^4\theta + \cos^4\theta) \\
H_{16} &= (C_{11} - C_{12} - 2C_{66})\sin\theta\cos^3\theta + (C_{12} - C_{22} + 2C_{66})\sin^3\theta\cos\theta \\
H_{22} &= C_{11}\sin^4\theta + 2(C_{12} + 2C_{66})\sin^2\theta\cos^2\theta + C_{22}\cos^4\theta \\
H_{26} &= (C_{11} - C_{22} - 2C_{66})\sin^3\theta\cos\theta + (C_{12} - C_{22} + 2C_{66})\sin\theta\cos^3\theta \\
H_{66} &= (C_{11} + C_{22} - 2C_{12} - 2C_{66})\sin^2\theta\cos^2\theta + C_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned}\right\} \tag{3}$$

式中，θ 为铺层主方向 x' 与 x 轴的夹角；

$$\left.\begin{aligned}
C_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad C_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\
C_{12} &= \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{12}E_1}{1 - \nu_{12}\nu_{21}}, \quad C_{66} = G_{12}
\end{aligned}\right\} \tag{4}$$

其中，E_1、E_2 为铺层主方向的弹性模量；ν_{12}、ν_{21} 为铺层主方向的泊松比；G_{12} 为铺层 $x' - y'$ 面内的剪切弹性模量。

设层合板屈曲时，中面面内沿 x、y 方向的位移为 $u_0(x,y)$、$v_0(x,y)$，中面的横向位移为 $w(x,y)$，层合板一点的应变则为

$$\left.\begin{aligned}
\varepsilon_1 &= \varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \\
\varepsilon_2 &= \varepsilon_y = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \\
\varepsilon_6 &= \gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x\partial y}
\end{aligned}\right\} \tag{5}$$

简写为

$$\{\varepsilon\} = \{\varepsilon_0\} + z\{S\} \tag{6}$$

式中：

$$\{\varepsilon_0\} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \qquad \{S\} = \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x\partial y} \end{Bmatrix} \tag{7}$$

分别为层合板中面的面内应变、中面的曲率和扭曲率。

如图 2 所示,根据薄膜力和弯矩的定义

$$\left.\begin{aligned} N &= \int_{-h/2}^{h/2} \sigma \mathrm{d}z \\ M &= \int_{-h/2}^{h/2} \sigma z \mathrm{d}z \end{aligned}\right\}$$

并利用式(2)及(5)、(6)及式(7)则得

$$\left.\begin{aligned} N_i &= \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} H_{ij}^{(k)} (\varepsilon_j^0 + zS_j) \mathrm{d}z \\ N_i &= \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} H_{ij}^{(k)} (z\varepsilon_j^0 + z^2 S_j) \mathrm{d}z \end{aligned}\right\} \tag{8}$$

积分得

$$\left.\begin{aligned} N_i &= A_{ij}\varepsilon_j^0 + B_{ij}S_j \\ M_i &= B_{ij}\varepsilon_j^0 + D_{ij}S_j \end{aligned} \quad (i,j = 1,2,6)\right\} \tag{9}$$

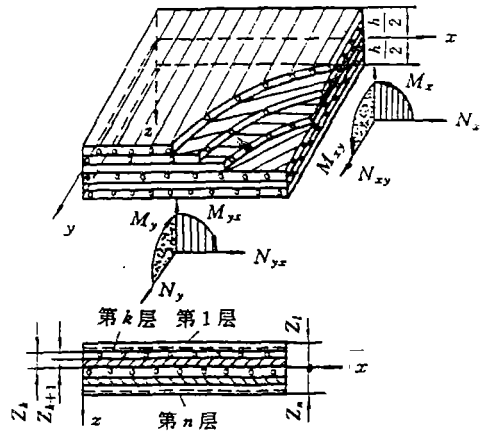

图 2　层合板的几何特征和坐标系

写成矩阵形式为

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ S \end{Bmatrix} \tag{10}$$

式中

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \quad [B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}, \quad [D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \tag{11}$$

为层合板屈曲的抗拉(压)和抗弯刚度矩阵。矩阵中的每项元素为

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{n} H_{ij}^{(k)} (z_{k+1} - z_k) \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} H_{ij}^{(k)} (z_{k+1}^2 - z_k^2) \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} H_{ij}^{(k)} (z_{k+1}^3 - z_k^3) \end{aligned}\right\} \tag{12}$$

引入力函数 ψ,使得

$$\left.\begin{aligned} N_x &= \frac{\partial^2 \psi}{\partial y^2} \\ N_y &= \frac{\partial^2 \psi}{\partial x^2} \\ N_{xy} &= -\frac{\partial^2 \psi}{\partial x \partial y} \end{aligned}\right\} \tag{13}$$

则层合板在横向荷载 $q(x,y)$ 作用下屈曲的控制方程为

$$\left.\begin{aligned} & D_{11} \frac{\partial^4 W}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 W}{\partial y^4} \\ & \quad = q + \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 W}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} - 2 \frac{\partial^2 W}{\partial x \partial y} \frac{\partial^2 \psi}{\partial x \partial y} \\ & A_{22}^* \frac{\partial^4 W}{\partial x^4} + (2A_{12}^* + A_{66}^*) \frac{\partial^4 W}{\partial x^2 \partial y^2} + A_{11}^* \frac{\partial^4 W}{\partial y^4} \\ & \quad = \left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \end{aligned}\right\} \tag{14}$$

式中，W 为挠度，A_{ij}^* 为抗拉(压)刚度矩阵的逆矩阵的元素。

四边简支层合板的边界条件为

$$\left.\begin{aligned}&W(0,y)=W(a,y)=M_x(0,y)=M_x(a,y)=0\\&W(x,0)=W(x,b)=M_y(x,0)=M_y(x,b)=0\\&\frac{1}{b}\int_0^b N_x\mathrm{d}y=\overline{N}_x,\quad \frac{1}{a}\int_0^a N_y\mathrm{d}x=\overline{N}_y\\&u(a,y)-u(0,y)=\delta_x,\quad v(x,b)-v(x,0)=\delta_y\end{aligned}\right\}\tag{15}$$

根据式(9)的第二式和式(13)，边界条件(15)可由 W 和 ψ 表示成

$$\left.\begin{aligned}&W(0,y)=W(a,y)=\frac{\partial^2W}{\partial x^2}(0,y)=\frac{\partial^2W}{\partial x^2}(a,y)=0\\&W(x,0)=W(x,b)=\frac{\partial^2W}{\partial y^2}(x,0)=\frac{\partial^2W}{\partial y^2}(x,b)=0\\&\frac{1}{b}\int_0^b\frac{\partial^2\psi}{\partial y^2}\mathrm{d}y=\overline{N}_x\quad,\quad \frac{1}{a}\int_0^a\frac{\partial^2\psi}{\partial x^2}\mathrm{d}x=\overline{N}_y\\&\int_0^a(A_{11}^*\frac{\partial^2\psi}{\partial y^2}+A_{12}^*\frac{\partial^2\psi}{\partial x^2}-\frac{1}{2}(\frac{\partial W}{\partial x})^2)\mathrm{d}x=\delta_x\\&\int_0^b(A_{12}^*\frac{\partial^2\psi}{\partial y^2}+A_{22}^*\frac{\partial^2\psi}{\partial x^2}-\frac{1}{2}(\frac{\partial W}{\partial y})^2\mathrm{d}y=\delta_y\end{aligned}\right\}\tag{16}$$

式中，$\overline{N}_x$ 和 $\overline{N}_y$ 为单位长度板受到的 x 轴和 y 轴方向的平均法向合力，δ_x 和 δ_y 为板的 x 和 y 轴方向的伸长。

2 层合板屈曲的边界元解法

为了满足上述边界条件，选取层合板的挠度函数、力函数和横向荷载函数分别以双重傅立叶级数形式表示为

$$\left.\begin{aligned}&W(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&\psi=\frac{N_x}{2}y^2+\frac{N_y}{2}x^2+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\\&q=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\end{aligned}\right\}\tag{17}$$

式中：

$$q_{mn}=\frac{4}{ab}\int_0^b\int_0^a q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y\tag{18}$$

本文在文献[2]的基础上，将基本解 $W^*(x,y)$ 作为权函数乘方程(14)的等号两边，然后对全域进行积分，再应用格林(Green)公式将区域积分转换为边界积分，则得到如下形式的边界积分方程

$$C_mW(x_m,y_m)=\oint_\Gamma[R_n(W)W^*-M_n(W)\frac{\partial W^*}{\partial n}+\frac{\partial W}{\partial n}M_n(W^*)-WR_n(W^*)]\mathrm{d}\Gamma+$$
$$\sum_{i=1}^{e}\Delta_iM_\tau(W^*)W_i-\sum_{i=1}^{e}\Delta_iM_\tau(W)W_i^*-\iint_\Gamma\bar{q}W^*\mathrm{d}\Gamma\tag{19}$$

式中：

$$C_m = \begin{cases} \theta_m/2\pi & \forall\ (x_m, y_m) \in \Gamma \\ 0 & \forall\ (x_m, y_m) \bar{\in} \Gamma \end{cases}$$

Γ 为层合板的边界，$M_n(W)$、$M_n(W^*)$、$M_\tau(W)$、$M_\tau(W^*)$、$R_n(W)$、$R_n(W^*)$ 分别为对应于法向弯矩、扭矩、等效应力的微分算子，$\Delta_i M_\tau(W)$、$\Delta_i M_\tau(W^*)$ 为角点上扭矩之跳跃值，q 为等效荷载，e 为边界上的角点数，θ_m 为边界点两切线的夹角。

为了寻找层合板屈曲问题的基本解，令

$$\bar{q} = N_x \frac{\partial^2 W}{\partial x^2} + 2N_{xy} \frac{\partial^2 W}{\partial x \partial y} + N_y \frac{\partial^2 W}{\partial y^2} \tag{20}$$

因此，方程(19)中的 $\iint_\Gamma \bar{q} W^* \mathrm{d}\Gamma$ 可表示为

$$\iint_\Gamma \bar{q} W^* \mathrm{d}\Gamma = \iint_\Gamma (N_x \frac{\partial^2 W}{\partial x^2} + 2N_{xy} \frac{\partial^2 W}{\partial x \partial y} + N_y \frac{\partial^2 W}{\partial y^2}) W^* \mathrm{d}\Gamma$$

若将域面积划成 m 份，取每份中点值 $(\frac{\partial^2 W}{\partial x^2})_i$，$(\frac{\partial^2 W}{\partial x \partial y})_i$，$(\frac{\partial^2 W}{\partial y^2})_i$ 为此块面积内的相应值，则上式变为

$$\iint_\Gamma \bar{q} W^* \mathrm{d}\Gamma = \sum_{i=1}^{m} [(\iint_{\Gamma_i} N_x W^* \mathrm{d}\Gamma)(\frac{\partial^2 W}{\partial x^2})_i + (2\iint_{\Gamma_i} N_{xy} W^* \mathrm{d}\Gamma)(\frac{\partial^2 W}{\partial x \partial y})_i + (\iint_{\Gamma_i} N_y W^* \mathrm{d}\Gamma)(\frac{\partial^2 W}{\partial y^2})_i] \tag{21}$$

将边界 Γ 划分成 n 个边界元，采用边界位移按二次等参数边界元，则离散化的边界积分方程为

$$\begin{aligned} C_m W(x_m, y_m) = & \sum_{i=1}^{n_j} [(R_n(W))_i \int_{\Gamma_{ij}} W^* \mathrm{d}\Gamma - (M_n(W))_i \int_{\Gamma_{ij}} \frac{\partial W^*}{\partial n} \mathrm{d}\Gamma + \\ & (\frac{\partial W}{\partial n})_i \int_{\Gamma_{ij}} M_n(W^*) \mathrm{d}\Gamma - W_i \int_{\Gamma_{ij}} R_n(W^*) \mathrm{d}\Gamma] + \sum_{i=1}^{e} \Delta_i M_\tau(W^*) W_i - \\ & \sum_{i=1}^{e} \Delta_i M_\tau(W) W_i^* - \sum_{i=1}^{n_j} [(\iint_{\Gamma_{ij}} N_x W^* \mathrm{d}\Gamma)(\frac{\partial^2 W}{\partial x^2})_{ij} + \\ & (2\iint_{\Gamma_{ij}} N_{xy} W^* \mathrm{d}\Gamma)(\frac{\partial^2 W}{\partial x \partial y})_{ij} + (\iint_{\Gamma_{ij}} N_y W^*)(\frac{\partial^2 W}{\partial y^2})_{ij}] \end{aligned} \tag{22}$$

则整个层合板屈曲时的边界积分方程用矩阵表示为

$$[A]_{2n \times 2n} \{X\}_{2n \times 1} = [B]_{2n \times 3m} \{Y\}_{3m \times 1} \tag{23}$$

式中 $[A]$ 为系数矩阵，$\{X\}$ 为边界未知量列阵，$\{Y\}$ 为域面积元位移二阶导数列阵，$[B]$ 为相应的系数矩阵。

为确定未知量 $\{Y\}$，再利用方程(19)求域内点未知量 $\{Y\}$。对域内每个面积元列出边界积分方程，用矩阵表示为

$$\{Y\}_{3m \times 1} + [A_1]_{3m \times 2n} \{X\}_{2n \times 1} = [B_1]_{2m \times 3m} \{Y\}_{3m \times 1} \tag{24}$$

由方程(23)有　　$\{X\} = [A]^{-1}[B]\{Y\}$

将其代入方程(24)，整理后得

$$\{Y\} = ([B_1] + [A_1][A]^{-1}[B])\{Y\} \tag{25}$$

这就是确定纤维增强复合材料层合板屈曲临界荷载的特征方程。

3 数值算例

用本文方法计算下列算例：

例 1 在正弦分布荷载作用下四边简支的纤维增强复合材料层合矩形板屈曲临界荷载及挠度的特征。

设

$$q = q_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{26}$$

由式(14)、(16)、(17)及式(18)可得 q_0 与层合板中心挠度 W_c 近似关系式为

$$q_0 = \frac{D_{11}b^4 + 2(D_{12} + 2D_{66})a^2b^2 + D_{22}a^4}{2a^4b^4}\pi^2 W_c + \frac{a^4A_{11}^* + b^4A_{22}^*}{14a^4b^4A_{11}^*A_{22}^*}\pi^4 W_c^3 \tag{27}$$

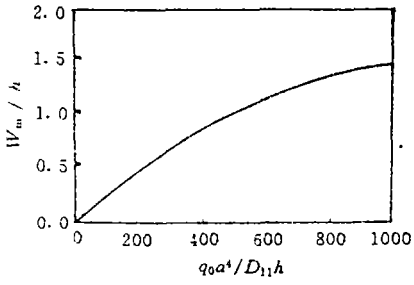

图 3 在正弦分布荷载作用下四边简支层合矩形板荷载与中心挠度关系曲线

以玻璃纤维/环氧复合材料层合矩形板为例，设其特征常量为

$a = 18.42\text{cm}$，$b = 12.28\text{cm}$，$h = 0.2536\text{cm}$

$A_{11} = A_{22} = 29.75 \times 10^6\text{N/m}$

$A_{12} = 7.325 \times 10^6\text{N/m}$

$A_{66} = 4.895 \times 10^6\text{N/m}$

$D_{11} = D_{22} = 14.26\text{N-m}$

$D_{12} = 3.340\text{N-m}$

$D_{66} = 2.162\text{N-m}$

则由式(27)及式(25)可得到层合板的无量纲临界荷载-中心挠度关系曲线如图 3 所示。

例 2 在均布荷载作用下四边简支相对位移 δ_x 和 δ_y 为零的纤维增强复合材料层合矩形板的临界荷载及挠度特征。

同理，由上述方法和公式得到层合板的荷载-中心挠度关系为

$$q = \frac{D_{11}b^4 + 2(D_{12} + 2D_{66})a^2b^2 + D_{22}a^4}{14a^4b^4} + \frac{3a^2b^2A_{11}^*A_{22}^* + 2(a^4A_{11}^* + b^4A_{22}^*)(A_{12}^{*2} - 3A_{11}^*A_{22}^*)}{238a^4b^4A_{11}^*A_{22}^*(A_{12}^{*2} - A_{11}^*A_{22}^*)}\pi^6 W_m^3 \tag{28}$$

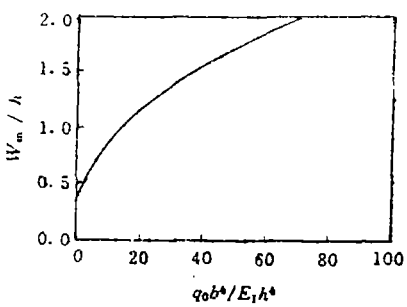

图 4 在均布荷载作用下四边简支层合矩形板荷载与中心挠度关系曲线

这里考虑碳纤维/环氧复合材料层合矩形板，其特征常数与例 1 同，其弹性常数为

$E_1 = 143.0 \times 10^9\text{N/m}^2$

$E_2 = E_3 = 16.23 \times 10^9\text{N/m}^2$

$G_{12} = G_{23} = G_{31} = 6.165 \times 10^9\text{N/m}^2$

$\nu_{12} = \nu_{23} = \nu_{31} = 0.23$

由式(4)决定 C_{ij} 后，通过数值计算得到三层[0°/90°/0°]层合板的无量纲临界荷载-中心挠度关系如图 4 所示。

4 结　论

根据用边界元法对纤维增强复合材料正交各向异性层合板屈曲性态的分析和算例计算结果，可得如下结论：

(1) 本文在边界元法中采用了双重傅立叶级数表示层合板的挠度函数、力函数及横向荷载函数和引入等效荷载概念进行分析的方法是可行和有效的，且具有足够精度，满足实际工程需要。

(2) 本文方法简便，计算工作量小，输入数据少，所编程序适宜在微机上实施。

(3) 从数值算例计算结果看出，当域面积单元数 m 及边界单元数 n 增大时，计算精度随之提高。

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ANALYSIS OF THE BUCKLING BEHAVIOUR OF FIBER-REINFORCED COMPOSITE MATERIAL LAMINATED PLATES BY BOUNDARY ELEMENT METHOD

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Abstract　The buckling behaviour of orthotropic fiber-reinforced composite laminates is analyzed by the boundary element method. In order to overcome the difficulties in finding the fundamental solution to the buckling of orthotropic laminates in boundary element method, the double Fourier series are used and a concept of equivalent load is introduced. The eigen equations of the critical loads of the laminated plates are determined. The computed examples show that the method is reliable and efficient.

Key words　composite material laminated plates, boundary element method, fundamental solution, buckling behaviour